

Classification of BPS Objects in $\mathcal{N} = 6$ Chern-Simons Matter Theory

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Abstract

We investigate BPS conditions preserving $n/12$ ($n = 1, \dots, 6$) supersymmetries in the Aharony-Bergman-Jafferis-Maldacena (ABJM) model. The BPS equations are classified in terms of the number of preserved supercharges and remaining subgroups of the $SU(4)_R$ symmetry. We study structures of a map between projection conditions for the supercharges in eleven dimensions and those in the ABJM model. The BPS configurations in the ABJM model can be interpreted as known BPS objects in eleven-dimensional M-theory, such as intersecting M2, M5-branes, M-waves, KK-monopoles and M9-branes. We also show that these BPS conditions reduce to those in $\mathcal{N} = 8$ super Yang-Mills theory via the standard D2-reduction procedure in a consistent way with the M-theory interpretation of the BPS conditions.

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1 Introduction

In the past two years a great amount of effort has been invested in a new class of superconformal field theories in three dimensions. Especially two models have been extensively studied. The first is the three-dimensional $\mathcal{N} = 8$ superconformal field theory with an exotic three-algebra structure proposed by Bagger, Lambert and Gustavsson (BLG model) [1, 2] and the other is the three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons matter theory with $U(N) \times U(N)$ gauge symmetry proposed by Aharony-Bergman-Jafferis-Maldacena (ABJM model) [3]. The ABJM model with Chern-Simons level k is expected to describe the low-energy effective theory of N coincident M2-branes located on the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold fixed point. One way to investigate this conjecture is to study the BPS equations and study whether they admit BPS objects that can be identified with M-theoretical objects such as M2 and M5-branes and so on. Several configurations containing M2, M5 and other objects have been found to be solutions of the classical equations (BPS equations or equations of motion). These can be found, for example, in [4–10]. However the complete catalog of solutions of the above equations has not yet been obtained.

In the present paper, we make further investigation on BPS configurations in the ABJM model. Rather than finding explicit solutions of the BPS equations, here we focus on the classification of BPS conditions and the physical interpretations of the BPS objects. Similar analysis has been carried out both in the BLG and ABJM model [11–13]. In this viewpoint we show that co-dimension one and two BPS objects in ABJM model are systematically classified in terms of the number of remaining supersymmetry (SUSY) and the unbroken subgroup of the $SU(4)_R$ symmetry. We classify the BPS conditions by using a projection matrix \mathcal{A} which specifies the preserved supercharges via projection condition $\mathcal{A}\epsilon = \epsilon$ for supersymmetry transformation parameters ϵ . We also make an analysis on the physical interpretation of the BPS conditions by finding a map between the conditions in eleven dimensions and those in the ABJM model. We show that BPS objects in the ABJM model can be interpreted as M-theoretical objects such as M2-branes, M5-branes and so on. We also consider the reduction of the BPS conditions to ten dimensions by the novel Higgs mechanism [14]. We show that the BPS objects in the ABJM model are consistently reduced to those in the $\mathcal{N} = 8$ super Yang-Mills theory which are identified with various BPS objects in type IIA string theory.

The organization of this paper is as follows. In the next section, we briefly review the ABJM model and introduce conventions, notations that we use throughout this paper. In section 3, we classify the 1/2 BPS equations of the ABJM model in terms of supersymmetry projection matrices. In section 4, we discuss the supersymmetry projection conditions for the M2-branes and the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold from eleven-dimensional viewpoint. After that we relate the M-theoretical objects to the 1/2 BPS states in the ABJM model. In section 5, BPS configurations with less than 6 preserved supercharges are discussed. In section 6, we perform

the reduction of the BPS equations to the one in the $\mathcal{N} = 8$ super Yang-Mills theory through the Higgs mechanism and analyze the physical meaning of the solutions. Section 7 is conclusions and discussions. Detailed expressions of the BPS equations are presented in Appendix.

2 ABJM model

In this section, we briefly review the ABJM model and introduce our notations and conventions that we use throughout this paper. The ABJM model is a $(2+1)$ -dimensional $\mathcal{N} = 6$ Chern-Simons matter system that consists of $U(N) \times U(N)$ gauge fields A_μ, \hat{A}_μ with Chern-Simons levels k and $-k$, four complex scalar fields Y^A ($A = 1, 2, 3, 4$) and their superpartners ψ_A in the bifundamental representation of the gauge group¹. This model exhibits a manifest global $SU(4)_R \cong Spin(6)_R$ symmetry, for which Y^A and ψ_A are in the fundamental (upper indices) and antifundamental (lower indices) representations. This model is expected to describe the low-energy effective world-volume theory of N coincident M2-branes probing $\mathbb{C}^4/\mathbb{Z}_k$ in eleven dimensions. The bosonic part of the action is given as follows,

$$S = -\frac{k}{2\pi} \int d^3x \text{Tr} \left[D_\mu Y^A D^\mu Y_A^\dagger + \frac{2}{3} \Upsilon_A^{BC} (\Upsilon_A^{BC})^\dagger \right] + \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left[A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho - \hat{A}_\mu \partial_\nu \hat{A}_\rho - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho \right], \quad (2.1)$$

where the covariant derivative is defined as $D_\mu Y^A \equiv \partial_\mu Y^A + iA_\mu Y^A - iY^A \hat{A}_\mu$ and Υ_A^{BC} is given by

$$\Upsilon_A^{BC} \equiv Y^B Y_A^\dagger Y^C + \frac{1}{2} \delta_A^B \left(Y^C Y_D^\dagger Y^D - Y^D Y_D^\dagger Y^C \right) - (B \leftrightarrow C) \quad . \quad (2.2)$$

Hereafter, we will mainly discuss the supersymmetric variations of the fermions and BPS equations.

To obtain the BPS equations in the ABJM model, we consider the supersymmetric variation of the fermionic fields. In the notation with the manifest $SU(4)_R$ symmetry, they are given by [5, 16]

$$\delta\psi_A = \left(\gamma^\mu D_\mu Y^B \delta_A^C + \Upsilon_A^{BC} \right) (\Gamma_i)_{BC} \epsilon_i. \quad (2.3)$$

Here, ϵ_i ($i = 1, \dots, 6$) are $(2+1)$ -dimensional Majorana spinor parameters which are in the vector representation **6** of $SO(6)_R$ and Γ^i ($i = 1, \dots, 6$) are the matrices satisfying

$$\Gamma_i \Gamma_j^\dagger + \Gamma_j \Gamma_i^\dagger = 2\delta_{ij}, \quad (\Gamma_i)_{AB} = -(\Gamma_i)_{BA}, \quad \frac{1}{2} \epsilon^{ABCD} (\Gamma_i)_{CD} = (\Gamma_i^\dagger)^{AB}. \quad (2.4)$$

¹In this paper we basically employ the notations in [15], but slightly modified them such that the Chern-Simons level k appears as an overall coefficient of the action.

Throughout this paper, we will use the following explicit forms

$$\begin{aligned}\Gamma_1 &= \begin{pmatrix} \sigma_2 & \\ & \sigma_2 \end{pmatrix}, & \Gamma_2 &= \begin{pmatrix} -i\sigma_2 & \\ & i\sigma_2 \end{pmatrix}, & \Gamma_3 &= \begin{pmatrix} & i\sigma_2 \\ i\sigma_2 & \end{pmatrix}, \\ \Gamma_4 &= \begin{pmatrix} & i\sigma_1 \\ -i\sigma_1 & \end{pmatrix}, & \Gamma_5 &= \begin{pmatrix} & -i\sigma_3 \\ i\sigma_3 & \end{pmatrix}, & \Gamma_6 &= \begin{pmatrix} & -\mathbf{1}_2 \\ \mathbf{1}_2 & \end{pmatrix},\end{aligned}\quad (2.5)$$

where σ_i ($i = 1, 2, 3$) are the Pauli matrices. The $Spin(6)_R$ generators Σ_{ij} ($i, j = 1, \dots, 6$) are given in terms of Γ_i as

$$\Sigma_{ij} \equiv \frac{i}{4}(\Gamma_i^\dagger \Gamma_j - \Gamma_j^\dagger \Gamma_i). \quad (2.6)$$

For the $SO(2, 1)$ gamma matrices γ^μ ($\mu = 0, 1, 2$), we will use

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.7)$$

In this basis, the Majorana spinor condition for the spinor parameters ϵ_i is simply given by

$$\epsilon_i = \epsilon_i^*, \quad (2.8)$$

where ϵ_i^* denotes the complex conjugate of ϵ_i .

3 Classification of 1/2 BPS equations

3.1 Supersymmetry projection conditions

In this section, we classify 1/2 BPS equations which correspond to the most elementary BPS objects in the ABJM model. They are characterized by the preserved supercharges determined by the projection condition of the form

$$\gamma \Xi_{ij} \epsilon_j = \epsilon_i, \quad (i, j = 1, 2, \dots, 6). \quad (3.1)$$

Here γ is a 2-by-2 matrix acting on spinor indices and Ξ is a 6-by-6 matrix acting on $SO(6)_R$ vector indices. Once the matrices γ and Ξ are given, BPS equations can be obtained by requiring that the supersymmetric variations of the fermions (2.3) vanish for the spinor parameters ϵ_i specified by the condition (3.1).

Now, let us classify the supersymmetry projection matrix $\mathcal{A} \equiv \gamma \otimes \Xi$ acting on $(\mathbf{2}, \mathbf{6})$ of $SO(2, 1) \times SO(6)_R$ by imposing conditions which lead to half BPS equations. We assume that \mathcal{A} is traceless

$$\text{Tr } \mathcal{A} = 0, \quad (3.2)$$

where the trace is taken over $SO(2,1) \times SO(6)_R$ indices. We also assume that the square of \mathcal{A} is the identity map

$$\mathcal{A}^2 = (\gamma \otimes \Xi)^2 = \mathbf{1}_2 \otimes \mathbf{1}_6. \quad (3.3)$$

Furthermore, the matrix elements of γ and Ξ should be real since the spinor parameters ϵ_i are Majorana spinors satisfying $\epsilon_i^* = \epsilon_i$. For an operator \mathcal{A} satisfying these conditions, there are generically $12 \times \frac{1}{2} = 6$ linearly independent solutions to the condition (3.1).

First, let us classify the 2-by-2 real matrix γ , which can be written as a linear combination of the unit matrix and the gamma matrices $\gamma = a\mathbf{1}_2 + b^\mu \gamma_\mu$. By using $SO(2,1)$ Lorentz transformations, we can always fix the matrix γ as

$$\gamma = \begin{cases} a\mathbf{1}_2 + b\gamma_0 & \text{if } b^\mu b_\mu = -b^2 \\ a\mathbf{1}_2 + b\gamma_2 & \text{if } b^\mu b_\mu = b^2 \end{cases}, \quad (3.4)$$

where a and b are real parameters. From the condition (3.3), one finds that possible matrices are $\gamma = \mathbf{1}_2$, γ_0 or γ_2 (up to Lorentz transformation). However, we can show that BPS equations are trivial for the unit matrix $\gamma = \mathbf{1}_2$. Therefore, there are essentially two possibilities for the 2-by-2 matrix γ

$$\gamma = \gamma_0 \text{ or } \gamma_2. \quad (3.5)$$

Next, let us classify the 6-by-6 real matrix Ξ . Because of the condition (3.3), the matrix Ξ should satisfy

$$\Xi^2 = \begin{cases} -\mathbf{1}_6 & \text{if } \gamma = \gamma_0 \\ \mathbf{1}_6 & \text{if } \gamma = \gamma_2 \end{cases}. \quad (3.6)$$

If Ξ is an anti-symmetric matrix, its square has negative eigenvalues. Therefore, the 2-by-2 matrix γ should be γ_0 for an anti-symmetric matrix Ξ . In this case, the matrix Ξ satisfying $\Xi^2 = -\mathbf{1}_2$ can always be fixed in the following standard form by $SO(6)_R$ transformations

$$\Xi \rightarrow B \equiv \pm \left(\begin{array}{c|c|c} i\sigma_2 & & \\ \hline & i\sigma_2 & \\ \hline & & i\sigma_2 \end{array} \right). \quad (3.7)$$

On the other hand, if Ξ is a symmetric matrix, γ should be γ_2 since all the eigenvalues of Ξ^2 are positive. Any symmetric matrix Ξ satisfying $\Xi^2 = \mathbf{1}_6$ can always be diagonalized by $SO(6)_R$ transformations as

$$\Xi \rightarrow C^{(m,n)} \equiv \left(\begin{array}{c|c} \mathbf{1}_m & \\ \hline & -\mathbf{1}_n \end{array} \right), \quad m + n = 6. \quad (3.8)$$

In summary, the possible 1/2 BPS projection matrices can be fixed by using $SO(2, 1) \times SO(6)_R$ transformation in the following forms

$$\mathcal{A} = \begin{cases} \gamma_0 \otimes B & (3.9) \\ \gamma_2 \otimes C^{(m,n)} & (3.10) \end{cases}$$

Note that the second condition guarantees that the $\mathcal{N} = (m, n)$, ($m + n = 6$) supersymmetries are preserved in terms of $(1+1)$ -dimensional viewpoint ($x^{0,1}$ -directions). Later, we will see that this condition is relaxed to the $m + n \leq 6$ case when we consider configurations with less than 1/2 supersymmetries.

3.2 BPS equations

Let us derive the BPS equations corresponding to the 1/2 BPS projection matrices which we have determined above. First, we consider the case $\mathcal{A} = \gamma_0 \otimes B$. For the spinor parameters satisfying $\gamma_0 B_{ij} \epsilon_j = \epsilon_i$, the supersymmetric variations of the fermions are given by

$$\begin{aligned} \delta\psi_A &= [\gamma^0 D_0 Y^B (\Gamma_i)_{BA} + \Upsilon_A^{BC} (\Gamma_i)_{BC}] \epsilon_i + (\gamma^1 D_1 Y^B + \gamma^2 D_2 Y^B) (\Gamma_i)_{BA} \epsilon_i \\ &= [D_0 Y^B (\Gamma_j)_{BA} B_{ji} + \Upsilon_A^{BC} (\Gamma_i)_{BC}] \epsilon_i + [D_1 Y^B (\Gamma_i)_{BA} + D_2 Y^B (\Gamma_j)_{BA} B_{ji}] \gamma_1 \epsilon_i. \end{aligned} \quad (3.11)$$

Requiring $\delta\psi_A = 0$, we obtain the following 1/2 BPS equations

$$0 = D_0 Y^B (\Gamma_j)_{BA} B_{ji} + \Upsilon_A^{BC} (\Gamma_i)_{BC}, \quad (3.12)$$

$$0 = D_1 Y^B (\Gamma_i)_{BA} - D_2 Y^B (\Gamma_j)_{BA} B_{ji}. \quad (3.13)$$

The more explicit forms of these equations are given in Appendix A (see (A.17)-(A.19)). These equations are compatible with the equations of motion provided that the following Gauss' law equations are satisfied

$$\begin{aligned} 0 &= \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} + i [Y^A (D^\rho Y^A)^\dagger - (D^\rho Y^A) Y^{A\dagger}], \\ 0 &= \frac{1}{2} \epsilon^{\mu\nu\rho} \hat{F}_{\mu\nu} + i [(D^\rho Y^A)^\dagger Y^A - Y^{A\dagger} (D^\rho Y^A)]. \end{aligned} \quad (3.14)$$

These BPS equations are equivalent to those discussed in [8]. A solution to these equations contains co-dimension two vortex type objects studied in [6, 18, 19].

Next, let us consider the case $\mathcal{A} = \gamma_2 \otimes C^{(m,n)}$. For the spinor parameters satisfying $\gamma_2 C_{ij}^{(m,n)} \epsilon_j = \epsilon_i$, the supersymmetric variations for the fermions are given by

$$\begin{aligned} \delta\psi_A &= [\gamma^2 D_2 Y^B (\Gamma_i)_{BA} + \Upsilon_A^{BC} (\Gamma_i)_{BC}] \epsilon_i + (\gamma^0 D_0 Y^B + \gamma^1 D_1 Y^B) (\Gamma_i)_{BA} \epsilon_i \\ &= [D_2 Y^B (\Gamma_j)_{BA} C_{ji}^{(m,n)} + \Upsilon_A^{BC} (\Gamma_i)_{BC}] \epsilon_i + [D_0 Y^B (\Gamma_i)_{BA} - D_1 Y^B (\Gamma_j)_{BA} C_{ji}^{(m,n)}] \gamma_0 \epsilon_i. \end{aligned} \quad (3.15)$$

From the condition $\delta\psi_A = 0$, we obtain the following 1/2 BPS equations

$$0 = D_2 Y^B (\Gamma_j)_{BA} C_{ji}^{(m,n)} + \Upsilon_A^{BC} (\Gamma_i)_{BC}, \quad (3.16)$$

$$0 = D_0 Y^B (\Gamma_i)_{BA} - D_1 Y^B (\Gamma_j)_{BA} C_{ji}^{(m,n)}. \quad (3.17)$$

The more explicit forms of these equations are given in Appendix A. In addition to these equations, the Gauss' law equations (3.14) should also be satisfied. Examples of solution to these equations contain co-dimension one fuzzy funnels and domain walls in the massless and massive ABJM models which have been investigated in [4, 5].

It is worthwhile to note that the discussions on the BPS conditions in this paper can be generalized to the massive ABJM model which keeps maximal supersymmetry [17]. This is achieved by the following replacement in the BPS conditions,

$$\Upsilon_A^{BC} \rightarrow \Upsilon_A^{BC} + \frac{1}{2} M_A^{[B} Y^{C]}, \quad M_A^B = m \text{diag}(1, 1, -1 - 1), \quad (3.18)$$

where m is a mass parameter.

4 BPS conditions from eleven-dimensional viewpoint

4.1 Supersymmetry preserved by M2-branes and orbifold

In the previous section, we have classified 1/2 BPS equations in the ABJM model. To understand the physical meaning of the BPS conditions in the M2-brane world-volume theory, it is useful to analyze a map between BPS projection matrices in eleven-dimensional space-time $\mathbb{R}^{(2,1)} \times \mathbb{C}^4/\mathbb{Z}_k$ and their counterparts in the ABJM model.

First, let us specify the supersymmetry preserved by the orbifold in eleven dimensions and N coincident M2-branes located at the orbifold fixed point. Let x^M ($M = 0, 1, \dots, 10$) be the eleven-dimensional space-time coordinates and y^A ($A = 1, 2, 3, 4$) be the complex coordinates of \mathbb{C}^4 defined by

$$y^A = x^{2A+1} + ix^{2A+2}. \quad (4.1)$$

The world-volume coordinate x^μ is chosen so that x^μ ($\mu = 0, 1, 2$) is identified with the space-time coordinate x^M ($M = 0, 1, 2$) while the transverse directions y^A are identified with the bi-fundamental fields Y^A living on the world-volume. The \mathbb{Z}_k orbifold action on the transverse coordinates is defined by

$$y^A \sim e^{2\pi i \frac{n}{k}} y^A, \quad (n = 1, 2, \dots, k). \quad (4.2)$$

In the presence of N M2-branes extending along $x^{0,1,2}$ -directions, the preserved supercharges (16 SUSY) are specified by

$$\hat{\Gamma}_{012} \xi = \xi, \quad (4.3)$$

where ξ is an eleven-dimensional Majorana spinor parameter with 32 components and $\hat{\Gamma}_M$ ($M = 0, 1, \dots, 10$) are eleven-dimensional gamma matrices satisfying $\{\hat{\Gamma}_M, \hat{\Gamma}_N\} = 2\eta_{MN}$. The Lorentz symmetry $SO(10, 1)$ is broken by the M2-branes to $SO(2, 1) \times SO(8)$, under which the spinor parameters satisfying (4.3) transform as $(\mathbf{2}, \mathbf{8}_c)$.

In addition to the condition (4.3), the supersymmetric transformation parameter is further projected by the \mathbb{Z}_k orbifold action. The \mathbb{Z}_k orbifold action on \mathbb{C}^4 , which is a subgroup of $SO(8)$, is defined by (4.2) and the corresponding group elements acting on spinors are

$$\exp\left(\frac{2\pi n}{k}J\right) \in Spin(8), \quad J \equiv i \sum_{A=1}^4 \hat{\Sigma}_{2A+1, 2A+2}, \quad (n = 1, \dots, k). \quad (4.4)$$

Here, $\hat{\Sigma}_{IJ}$ ($I, J = 3, \dots, 10$) are the $so(8)$ generators in the spinor representation defined by

$$\hat{\Sigma}_{IJ} = \frac{i}{4}[\hat{\Gamma}_I, \hat{\Gamma}_J]. \quad (4.5)$$

The supercharges preserved in the orbifold are those which are invariant under the orbifold action

$$\exp\left(\frac{2\pi n}{k}J\right) \xi = \xi, \quad (n = 1, \dots, k). \quad (4.6)$$

The orbifold projection breaks the $SO(8)$ symmetry down to $SU(4) \times U(1)$ subgroup corresponding to the generators which commutes with J . This $SU(4)$ subgroup is nothing but the $SU(4)_R$ symmetry in the ABJM model. Explicitly, the generators of $su(4) \oplus u(1)$ subalgebra are given by

$$S^A{}_B \equiv -\frac{1}{2}[\hat{\Gamma}_+^A, \hat{\Gamma}_{-B}], \quad (4.7)$$

where $\hat{\Gamma}_+^A$ and $\hat{\Gamma}_{-A}$ ($A = 1, 2, 3, 4$) are raising and lowering operators

$$\hat{\Gamma}_+^A \equiv \frac{\hat{\Gamma}_{2A+1} + i\hat{\Gamma}_{2A+2}}{2}, \quad \hat{\Gamma}_{-A} \equiv \frac{\hat{\Gamma}_{2A+1} - i\hat{\Gamma}_{2A+2}}{2}. \quad (4.8)$$

Under the breaking of the symmetry $SO(8) \rightarrow SU(4) \times U(1)$, the representation $\mathbf{8}_c$ of $SO(8)$ decomposes into the following representations of $SU(4)$,

$$\mathbf{8}_c \rightarrow \mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2}, \quad (4.9)$$

where the subscript indicates the charge under the $U(1)$ generator $-iJ = S^A{}_A$. Therefore, the components of the spinor ξ which are invariant under the orbifold action form a basis of the vector space $(\mathbf{2}, \mathbf{6})$ of $SO(2, 1) \times SU(4)$ ², which is spanned by the following normalized spinors

$$\begin{aligned} &|\alpha; \downarrow\downarrow\uparrow\uparrow\rangle, \quad |\alpha; \uparrow\uparrow\downarrow\downarrow\rangle, \quad |\alpha; \uparrow\downarrow\downarrow\uparrow\rangle, \\ &|\alpha; \downarrow\uparrow\uparrow\downarrow\rangle, \quad |\alpha; \downarrow\uparrow\downarrow\uparrow\rangle, \quad |\alpha; \uparrow\downarrow\uparrow\downarrow\rangle, \end{aligned} \quad (4.10)$$

² Note that in the case of $k = 1, 2$ the $\mathbf{1}_2 \oplus \mathbf{1}_{-2}$ components also satisfy Eq. (4.6), so that $(\mathbf{2}, \mathbf{8}_c)$ is preserved by the orbifold action.

while $(\mathbf{2}, \mathbf{1}_2 \oplus \mathbf{1}_{-2})$ part is spanned by the following spinors

$$|\alpha; \uparrow\uparrow\uparrow\uparrow\rangle, \quad |\alpha; \downarrow\downarrow\downarrow\downarrow\rangle, \quad (4.11)$$

where $\alpha = 1, 2$ is an index of $(2+1)$ -dimensional spinor and each arrow indicates the eigenvalue $\pm\frac{1}{2}$ of $\hat{\Sigma}_{2A+1, 2A+2}$. Namely, 16 supercharges specified by (4.3) is reduced to 12 SUSY expressed by (4.10). This is identified with the $\mathcal{N} = 6$ supersymmetry in the ABJM model. Throughout this paper, we will use the following orthonormal basis for the vector space $(\mathbf{2}, \mathbf{6})$ of $SO(2, 1) \times SU(4)$

$$\begin{aligned} \psi^{(\alpha,1)} &= \frac{|\alpha; \downarrow\downarrow\uparrow\uparrow\rangle + |\alpha; \uparrow\uparrow\downarrow\downarrow\rangle}{\sqrt{2}}, & \psi^{(\alpha,2)} &= i \frac{|\alpha; \downarrow\downarrow\uparrow\uparrow\rangle - |\alpha; \uparrow\uparrow\downarrow\downarrow\rangle}{\sqrt{2}}, \\ \psi^{(\alpha,3)} &= \frac{|\alpha; \uparrow\downarrow\downarrow\uparrow\rangle + |\alpha; \downarrow\uparrow\uparrow\downarrow\rangle}{\sqrt{2}}, & \psi^{(\alpha,4)} &= i \frac{|\alpha; \uparrow\downarrow\downarrow\uparrow\rangle - |\alpha; \downarrow\uparrow\uparrow\downarrow\rangle}{\sqrt{2}}, \\ \psi^{(\alpha,5)} &= \frac{|\alpha; \downarrow\uparrow\downarrow\uparrow\rangle + |\alpha; \uparrow\downarrow\uparrow\downarrow\rangle}{\sqrt{2}}, & \psi^{(\alpha,6)} &= i \frac{|\alpha; \downarrow\uparrow\downarrow\uparrow\rangle - |\alpha; \uparrow\downarrow\uparrow\downarrow\rangle}{\sqrt{2}}. \end{aligned} \quad (4.12)$$

For this choice of the basis of $(\mathbf{2}, \mathbf{6})$, the elements of $SO(6) \cong SU(4)/\mathbb{Z}_2$ are expressed by orthogonal matrices $O^T O = \mathbf{1}_6$.

The eleven-dimensional spinor parameter ξ can be projected onto $(\mathbf{2}, \mathbf{6})$ by using a projection operator P (12-by-32 matrix) which has the following structure

$$P^{(\alpha,i)}_{\hat{\alpha}} \equiv (\psi^{(\alpha,i)})_{\hat{\alpha}}^{\dagger}, \quad (\alpha = 1, 2, i = 1, \dots, 6), \quad (\hat{\alpha} = 1, \dots, 32). \quad (4.13)$$

where $\hat{\alpha}$ is the index of the eleven-dimensional spinor. This projection operator satisfies

$$PP^{\dagger} = \mathbf{1}_2 \otimes \mathbf{1}_6, \quad PS^A_B = (\mathbf{1}_2 \otimes \mathcal{S}^A_B)P, \quad PJ = 0, \quad (4.14)$$

where $(\mathcal{S}^A_B)_{ij} \equiv i(\Sigma_{ij})^A_B$ are the generators of $su(4) \cong so(6)$ in the vector representation $\mathbf{6}$. The first equation follows from the orthonormality of the basis $\{\psi^{(\alpha,i)}\}$. On the other hand, the second and the third equations, which indicate the transformation property of P under $SU(4)_R \times U(1)$, follow from the fact that the spinors $\psi^{(\alpha,i)}$ are in $\mathbf{6}_0$ of $SU(4) \times U(1)$.

For later convenience, let us define a projection operator \tilde{P} (4-by-32 matrix) which projects the 32-component spinor ξ onto $(\mathbf{2}, \mathbf{1}_2 \oplus \mathbf{1}_{-2})$

$$\tilde{P}^{(\alpha,i)}_{\hat{\alpha}} \equiv (\tilde{\psi}^{(\alpha,i)})_{\hat{\alpha}}^{\dagger}, \quad (\alpha = 1, 2, i = 1, 2), \quad (\hat{\alpha} = 1, \dots, 32), \quad (4.15)$$

where $\tilde{\psi}^{(\alpha,i)}$ are the basis of $(\mathbf{2}, \mathbf{1}_2 \oplus \mathbf{1}_{-2})$ given by

$$\tilde{\psi}^{(\alpha,1)} = \frac{|\alpha; \uparrow\uparrow\uparrow\uparrow\rangle + |\alpha; \downarrow\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}, \quad \tilde{\psi}^{(\alpha,2)} = i \frac{|\alpha; \uparrow\uparrow\uparrow\uparrow\rangle - |\alpha; \downarrow\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}. \quad (4.16)$$

The projection operator \tilde{P} has the following properties

$$\tilde{P}P^{\dagger} = 0, \quad \tilde{P}\hat{\Gamma}_{012} = 0, \quad \tilde{P}\tilde{P}^{\dagger} = \mathbf{1}_2 \otimes \mathbf{1}_2, \quad (4.17)$$

$$\tilde{P}(S^A_B + \frac{i}{4}\delta^A_B J) = 0, \quad \tilde{P}J = (\mathbf{1}_2 \otimes 2i\sigma_2)\tilde{P}. \quad (4.18)$$

Since the four-dimensional linear space $(\mathbf{2}, \mathbf{1}_2 \oplus \mathbf{1}_{-2})$ is the orthogonal complement of $(\mathbf{2}, \mathbf{6}_0)$ in $(\mathbf{2}, \mathbf{8}_c)$, the following identity holds

$$P^\dagger P + \tilde{P}^\dagger \tilde{P} = \frac{\mathbf{1}_{32} + \hat{\Gamma}_{012}}{2}. \quad (4.19)$$

4.2 1/2 BPS supersymmetry projection conditions from eleven dimensions

So far, we have seen that the 12 supercharges preserved by the M2-branes and orbifold are in $(\mathbf{2}, \mathbf{6})$ of $SO(2, 1) \times SU(4)$. They are specified by the conditions

$$\hat{\Gamma}_{012} \xi = \xi, \quad \exp\left(\frac{2\pi n}{k} J\right) \xi = \xi. \quad (4.20)$$

Here, the first condition has been imposed by the existence of M2-branes extending along $x^{0,1,2}$ -directions and the second condition is by the orbifold. The supersymmetric transformation parameter $\epsilon \in (\mathbf{2}, \mathbf{6})$ in the ABJM model is given by

$$\epsilon = P \xi. \quad (4.21)$$

In the following, we will observe that the BPS objects in the ABJM model are interpreted as M-theory branes in $\mathbb{R}^{(2,1)} \times \mathbb{C}^4/\mathbb{Z}_k$ by analyzing a map between projection matrices for eleven-dimensional SUSY parameters and those in the ABJM model.

Let us consider a projection condition for the eleven-dimensional spinor which are specified by the following equation

$$\hat{\Gamma} \xi = \xi. \quad (4.22)$$

Here, we assume that the operator $\hat{\Gamma}$ is invariant under the charge conjugation since the spinor parameter ξ is a Majorana spinor. In addition, we assume that the operator $\hat{\Gamma}$ satisfies

$$[\hat{\Gamma}, \hat{\Gamma}_3 \hat{\Gamma}_4 \cdots \hat{\Gamma}_{10}] = 0, \quad (4.23)$$

since the spinor ξ satisfying (4.20) has a definite chirality as an $SO(8)$ spinor. Let us define maps f and \tilde{f} by

$$f : \hat{\Gamma} \longmapsto \mathcal{A} \equiv P \hat{\Gamma} P^\dagger, \quad \tilde{f} : \hat{\Gamma} \longmapsto \tilde{\mathcal{A}} \equiv \tilde{P} \hat{\Gamma} P^\dagger, \quad (4.24)$$

which map eleven-dimensional projection matrices $\hat{\Gamma}$ to those in the ABJM model. The operators $\mathcal{A} : (\mathbf{2}, \mathbf{6}) \rightarrow (\mathbf{2}, \mathbf{6})$ and $\tilde{\mathcal{A}} : (\mathbf{2}, \mathbf{6}) \rightarrow (\mathbf{2}, \mathbf{1} \oplus \mathbf{1})$ are 12-by-12 and 4-by-12 matrices respectively. Using these operators, the equation (4.22) can be translated into the following conditions for the spinor parameters $\epsilon = P \xi$,

$$\mathcal{A} \epsilon = \epsilon, \quad \tilde{\mathcal{A}} \epsilon = 0, \quad (4.25)$$

where we have used the condition (4.19) and the fact that $\tilde{P}\xi = 0$ for the spinor parameters satisfying (4.20). Now, let us look for BPS projection matrices in eleven dimensions which are mapped to the 1/2 BPS projection matrices in (2+1) dimensions discussed in section 3. Such matrices are determined from the condition that the matrices (4.24) satisfy

$$\mathcal{A}^2 = \mathbf{1}_2 \otimes \mathbf{1}_6, \quad \text{Tr } \mathcal{A} = 0, \quad \tilde{\mathcal{A}} = 0. \quad (4.26)$$

Note that the reality condition $\mathcal{A}^* = \mathcal{A}$ is automatically satisfied since the operators P , \tilde{P} and $\hat{\Gamma}$ are invariant under the charge conjugation. We assume that the operators $\hat{\Gamma}$ takes the form

$$\hat{\Gamma} = \frac{1}{p!} \omega_{M_1 M_2 \dots M_p} \hat{\Gamma}_{M_1 M_2 \dots M_p}, \quad (4.27)$$

where $\omega_{M_1 M_2 \dots M_p}$ is a real p -form in the eleven-dimensional space-time. By using the relation $\hat{\Gamma}_{012} \xi = \hat{\Gamma}_{34 \dots 10} \xi = \xi$, one finds that independent operators are given by

$$\hat{\Gamma} = \begin{cases} \hat{\Gamma}^{(0)} & \equiv \hat{\Gamma}_\mu \\ \hat{\Gamma}^{(2)} & \equiv \frac{1}{2} \omega_{IJ} \hat{\Gamma}_{\mu IJ} \\ \hat{\Gamma}^{(4)} & \equiv \frac{1}{4!} \omega_{IJKL} \hat{\Gamma}_{\mu IJKL} \end{cases} \quad (\mu = 0, 1, 2, \quad I, J, K, L = 3, \dots, 10). \quad (4.28)$$

Note that the condition (4.23) is satisfied only when $\hat{\Gamma}$ contains even number of Γ^I ($I = 3, \dots, 10$). For the operators of the form (4.28), the corresponding operators \mathcal{A} take the form

$$\mathcal{A} = \gamma_\mu \otimes \Xi, \quad (4.29)$$

where γ_μ is the (2+1)-dimensional gamma matrix given in (2.7) and Ξ is a 6-by-6 matrix acting on **6** of $SO(6)$. This form of the operator \mathcal{A} implies that the traceless condition $\text{Tr } \mathcal{A} = 0$ is automatically satisfied for arbitrary Ξ . Now, let us examine which types of operators satisfy the conditions (4.26) and what types of M-theoretical objects are described by the corresponding BPS equations for each case of (4.28).

- $\hat{\Gamma} = \hat{\Gamma}^{(0)}$

First, let us consider the case $\hat{\Gamma} = \hat{\Gamma}_\mu$. Since the square of the operator $\mathcal{A} = P \hat{\Gamma}_\mu P^\dagger$ is

$$\mathcal{A}^2 = P (\hat{\Gamma}_\mu)^2 P^\dagger, \quad (4.30)$$

the gamma matrix $\hat{\Gamma}_\mu$ should be in the spacelike direction because of the first condition in (4.26). Therefore, we can always set $\hat{\Gamma}_\mu = \hat{\Gamma}_2$ by using $SO(2, 1)$ transformations without loss of generality. Since the spinor parameter satisfies $\hat{\Gamma}_{012} \xi = \hat{\Gamma}_{34 \dots 10} \xi = \xi$, the following three conditions are equivalent

$$\hat{\Gamma}_2 \xi = \xi \quad \Longleftrightarrow \quad \hat{\Gamma}_{01} \xi = \xi \quad \Longleftrightarrow \quad \hat{\Gamma}_{0134 \dots 10} \xi = \xi. \quad (4.31)$$

The second condition implies the existence of wave-type solutions, called the gravitational Brinkman waves [20] or M-waves which have momenta in the x^1 -direction. The third condition corresponds to a BPS object with $(9+1)$ -dimensional world-volume that is called M9-brane [21].

Under the maps f and \tilde{f} , the operator $\hat{\Gamma} = \hat{\Gamma}_2$ reduces to

$$\mathcal{A} = \gamma_2 \otimes \mathbf{1}_6, \quad \tilde{\mathcal{A}} = 0. \quad (4.32)$$

This is just the $1/2$ BPS projection matrix in (3.10) with $m = 6$, $n = 0$. Therefore the M2-branes (we call this "fiducial M2-branes" on which we are considering the world-volume theory), M-waves and M9-branes can co-exist preserving half of 12 SUSY, see Table 1 for the configuration. BPS objects that correspond to the projection $\mathcal{A} = \gamma_2 \otimes \mathbf{1}_6$ contain co-dimension one solutions. This fact is consistent with the intersection rules of various M-theoretical objects obtained in the supersymmetry algebra [22].

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M9	•	•		•	•	•	•	•	•	•	•
M-wave	•	•									

Table 1: A possible configuration that corresponds to the condition (4.31). The black dot means that that directions are filled by the corresponding objects. Note that the dashed line indicates the fact that the \mathbb{Z}_k orbifold action acts on the $(3, 4)$, $(5, 6)$, $(7, 8)$, $(9, 10)$ planes separately.

• $\hat{\Gamma} = \hat{\Gamma}^{(2)}$

Next, let us consider the case $\hat{\Gamma} = \frac{1}{2}\omega_{IJ}\hat{\Gamma}_{\mu IJ}$. To find $\hat{\Gamma}$ for which the conditions (4.26) are satisfied, it is convenient to classify the anti-symmetric tensor ω_{IJ} ($I, J = 3, \dots, 10$) in terms of representations of $SU(4)_R$. The anti-symmetric tensor ω_{IJ} is in **28** of $SO(8)$, which decomposes into the following $SU(4) \times U(1) \cong SO(6) \times U(1)$ representations

$$\mathbf{28} \rightarrow \mathbf{6}_2 \oplus (\mathbf{15}_0 \oplus \mathbf{1}_0) \oplus \mathbf{6}_{-2}. \quad (4.33)$$

In this 28-dimensional linear space, the kernel of the map $\tilde{f} : \hat{\Gamma} \mapsto \tilde{\mathcal{A}}$ is given by

$$\text{Ker } \tilde{f} = \mathbf{15}_0 \oplus \mathbf{1}_0. \quad (4.34)$$

Therefore, the third condition in (4.26) is satisfied only when the 2-form ω_{IJ} belongs to $\mathbf{15}_0 \oplus \mathbf{1}_0$. Note that the representation $\mathbf{15}_0 \oplus \mathbf{1}_0$ corresponds to $(1, 1)$ -type 2-forms $\omega_{A\bar{B}}$, which are invariant under the orbifold action. The first condition in (4.26) is satisfied only when the gamma matrix

$\hat{\Gamma}_\mu$ is in the timelike direction. Therefore, the matrix $\hat{\Gamma} = \hat{\Gamma}^{(2)}$ satisfying the conditions (4.26) can always be set in the following form

$$\hat{\Gamma} = \frac{1}{2}\omega_{IJ}\hat{\Gamma}_{0IJ}, \quad (\omega_{IJ}dx^I \wedge dx^J = \omega_{A\bar{B}}dy^A \wedge d\bar{y}^B). \quad (4.35)$$

Since **15** is the adjoint representation of $SO(6)$, the corresponding matrix $\hat{\Gamma}$ is mapped to \mathcal{A} whose 6-by-6 part Ξ is an anti-symmetric matrix. On the other hand, the singlet part **1** does not affect both \mathcal{A} and $\tilde{\mathcal{A}}$ since the singlet is in the kernel of the map $f : \hat{\Gamma} \rightarrow \mathcal{A}$

$$\text{Ker } f = \mathbf{6}_2 \oplus \mathbf{1}_0 \oplus \mathbf{6}_{-2}. \quad (4.36)$$

By appropriately choosing the singlet part (trace part of $\omega_{A\bar{B}}$), the 2-form $\omega_{A\bar{B}}$ becomes a volume form of a complex plane $\mathbb{C} \subset \mathbb{C}^4$. Since the projection condition (4.35) indicates that the BPS object has $(2+1)$ -dimensional world-volume, we conclude that the operators of the form (4.35) correspond to M2-branes stretching along complex planes specified by the $(1,1)$ -type 2-form $\omega_{A\bar{B}}$.

By using the fact that the spinor parameter satisfies $\hat{\Gamma}_{012}\xi = \hat{\Gamma}_{34\dots 10}\xi = \xi$, we can show that the following two conditions are equivalent

$$\frac{1}{2}\omega_{IJ}\hat{\Gamma}_{0IJ}\xi = \xi \quad \Longleftrightarrow \quad \frac{1}{6!}\tilde{\omega}_{IJKLMNOP}\hat{\Gamma}_{0IJKLMNOP}\xi = -\xi, \quad (4.37)$$

where $\tilde{\omega} \equiv *\omega$ is the Hodge dual of ω with respect to the metric on $\mathbb{C}^4/\mathbb{Z}_k$. The second condition can be interpreted as KK monopoles [23, 24] extending along $(6+1)$ -dimensional world-volume determined by the volume form $\tilde{\omega}_{IJKLMNOP}$. This fact is again an indication of co-existence of two intersecting M2-branes and KK monopoles.

As an example, let us consider the matrix $\hat{\Gamma} = \pm\hat{\Gamma}_0\hat{\Gamma}_9\hat{\Gamma}_{10}$ corresponding to M2-branes extending along $x^{0,9,10}$ -directions. In this case, the projected matrices \mathcal{A} and $\tilde{\mathcal{A}}$ are given by

$$\mathcal{A} = \gamma_0 \otimes B, \quad \tilde{\mathcal{A}} = 0. \quad (4.38)$$

This is just the 1/2 BPS condition (3.9) in the ABJM model. Therefore, the intersecting M2-branes with KK-monopoles are described by the 1/2 BPS equations (3.12) and (3.13) (see Table 2 for a possible configuration).

• $\hat{\Gamma} = \hat{\Gamma}^{(4)}$

Next, we consider the case $\hat{\Gamma} = \frac{1}{4!}\omega_{IJKL}\Gamma_{\mu IJKL}$. As in the previous case, let us classify the 4-form in terms of the representation of $SU(4)_R$. The 4-form ω_{IJKL} is in $\mathbf{35} \oplus \mathbf{35}'$ of $SO(8)$, which decomposes into the following $SU(4) \times U(1)$ representations

$$\mathbf{35} \oplus \mathbf{35}' \rightarrow \mathbf{1}_4 \oplus (\mathbf{6}_2 \oplus \mathbf{10}_2) \oplus (\mathbf{1}_0 \oplus \mathbf{15}_0 \oplus \mathbf{20}_0) \oplus (\mathbf{6}_{-2} \oplus \overline{\mathbf{10}}_{-2}) \oplus \mathbf{1}_{-4}. \quad (4.39)$$

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M2	•									•	•
KK	•			•	•	•	•	•	•		

Table 2: Intersecting M2-branes and KK-monopoles.

The kernel of the map $\tilde{f} : \hat{\Gamma} \mapsto \tilde{\mathcal{A}}$ is given by

$$\text{Ker } \tilde{f} = \mathbf{1}_4 \oplus \mathbf{10}_2 \oplus (\mathbf{1}_0 \oplus \mathbf{15}_0 \oplus \mathbf{20}_0) \oplus \overline{\mathbf{10}}_{-2} \oplus \mathbf{1}_{-4}. \quad (4.40)$$

The third condition in (4.26) is satisfied only when the 4-form ω_{IJKL} is in $\text{Ker } \tilde{f}$. The first condition in (4.26) is satisfied only when the gamma matrix $\hat{\Gamma}_\mu$ is in the spacelike direction. Therefore, the matrix $\hat{\Gamma} = \hat{\Gamma}^{(4)}$ satisfying the conditions (4.26) can always be set in the following form

$$\hat{\Gamma}^{(4)} = \frac{1}{4!} \omega_{IJKL} \hat{\Gamma}_{IJKL}, \quad \omega \notin \mathbf{6}_2 \oplus \mathbf{6}_{-2}. \quad (4.41)$$

The kernel of the map $f : \hat{\Gamma} \mapsto \mathcal{A}$ is given by

$$\text{Ker } f = \mathbf{1}_4 \oplus (\mathbf{6}_2 \oplus \mathbf{10}_2) \oplus \mathbf{15}_0 \oplus (\mathbf{6}_{-2} \oplus \overline{\mathbf{10}}_{-2}) \oplus \mathbf{1}_{-4}. \quad (4.42)$$

The matrix $\hat{\Gamma}$ should be in $\text{Ker } \tilde{f}$ to be a 1/2 BPS projection matrix in the ABJM model and components in $\text{Ker } f$ are irrelevant for the matrix \mathcal{A} . This means that the matrix \mathcal{A} is determined by $\mathbf{1}_0 \oplus \mathbf{20}_0$ components, which correspond to the symmetric tensor representation of $SO(6)$. Therefore, 1/2 BPS objects specified by the operator (4.41) are described in the ABJM model by the 1/2 BPS equations with symmetric matrices Ξ , which can be fixed to $C^{(m,n)}$ by $SO(6)$ transformations as we have seen in section 3.

Let $\omega^{(m,n)} = \frac{1}{4!} \omega_{IJKL}^{(m,n)} dx^I \wedge dx^J \wedge dx^K \wedge dx^L$ be the 4-forms corresponding to the matrices $\Xi = C^{(m,n)}$. Assuming that $\omega^{(m,n)}$ is invariant under the $Spin(m) \times Spin(n) \subset SU(4)$ transformations, we can find the following examples³ of 4-forms

$$\omega^{(6,0)} = \frac{1}{16} dy^A \wedge dy^B \wedge d\bar{y}_A \wedge d\bar{y}_B, \quad (4.43)$$

$$\omega^{(5,1)} = \frac{1}{64} [\mathcal{J}_{AB} \mathcal{J}_{\bar{C}\bar{D}} - \delta_{A\bar{C}} \delta_{B\bar{D}} + \delta_{A\bar{D}} \delta_{B\bar{C}}] dy^A \wedge dy^B \wedge d\bar{y}^{\bar{C}} \wedge d\bar{y}^{\bar{D}}, \quad (4.44)$$

$$\omega^{(4,2)} = \frac{1}{4} dy^1 \wedge dy^3 \wedge d\bar{y}^1 \wedge d\bar{y}^3, \quad (4.45)$$

$$\omega^{(3,3)} = \text{Re } d\check{y}^1 \wedge \text{Re } d\check{y}^2 \wedge \text{Re } d\check{y}^3 \wedge \text{Re } d\check{y}^4, \quad (4.46)$$

³ Note that there exists an ambiguity since we can add elements of $\text{Ker } f \cap \text{Ker } \tilde{f}$ without changing the reduced operators \mathcal{A} and $\tilde{\mathcal{A}}$.

where $\mathcal{J} \equiv \Gamma_6$ is a $Spin(5) \cong USp(4)$ invariant tensor and \check{y}^A are coordinates rotated by $\exp\left(\frac{\pi i}{2}\Sigma_{16}\right) \in SU(4)_R$

$$\check{y}^A \equiv \left(e^{\frac{\pi i}{2}\Sigma_{16}} \cdot y\right)^A. \quad (4.47)$$

The other 4-forms $\omega^{(2,4)}$, $\omega^{(1,5)}$, $\omega^{(0,6)}$ can be obtained by flipping the signs of $\omega^{(4,2)}$, $\omega^{(5,1)}$, $\omega^{(6,0)}$ and using $SU(4)$ transformations. Since the spinor parameter satisfies $\hat{\Gamma}_{012}\xi = \xi$, the condition $\omega_{IJKL}\hat{\Gamma}_{2IJKL}\xi = \xi$ can be rewritten as

$$\omega_{IJKL}\hat{\Gamma}_{01IJKL}\xi = \xi. \quad (4.48)$$

The BPS objects corresponding to the operators (4.48) have $(5+1)$ -dimensional world-volume and are identified with M5-branes that share one world-volume direction with our fiducial M2-branes. If the 4-form ω is a calibration, it determines a calibrated submanifold of $\mathbb{C}^4/\mathbb{Z}_k$. A related work in the BLG model can be found in [25].

The 4-form $\omega^{(6,0)}$ is proportional to the square of a Kähler form on $\mathbb{C}^4/\mathbb{Z}_k$, whose calibrated submanifold is the four-dimensional complex submanifold of $\mathbb{C}^4/\mathbb{Z}_k$. Therefore, in addition to the M-waves and M9-branes given in Table 1, the 1/2 BPS equation with $m = 6$, $n = 0$ admits calibrated M5-branes. For $\omega^{(4,2)}$ and $\omega^{(3,3)}$, the operator $\hat{\Gamma}$ is given by

$$\hat{\Gamma} = \frac{1}{4!}\omega_{IJKL}\hat{\Gamma}_{2IJKL} = \begin{cases} \hat{\Gamma}_{23478} & \text{for } \omega = \omega^{(4,2)} \\ \hat{\Gamma}_{23579} & \text{for } \omega = \omega^{(3,3)} \end{cases}. \quad (4.49)$$

Therefore, the BPS equations (3.16) and (3.17) with $\mathcal{N} = (4, 2)$ and $(3, 3)$ describe the configurations of the fiducial M2-branes ending on M5-branes (see Table 3 and 4). Note that because of the conditions $\hat{\Gamma}_{012}\xi = \hat{\Gamma}_{34\dots 10}\xi = \xi$, the following two conditions are equivalent

$$\frac{1}{4!}\omega_{IJKL}\hat{\Gamma}_{2IJKL}\xi = \xi \quad \Longleftrightarrow \quad \frac{1}{4!}\tilde{\omega}_{IJKL}\hat{\Gamma}_{2IJKL}\xi = \xi, \quad (4.50)$$

where $\tilde{\omega} \equiv *\omega$ is the Hodge dual of ω with respect to the metric on $\mathbb{C}^4/\mathbb{Z}_k$.

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M5	•	•		•	•			•	•		
M5	•	•				•	•			•	•

Table 3: M2-M5 configuration : $(m, n) = (4, 2)$.

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M5	•	•		•		•		•		•	
M5	•	•			•		•		•		•

Table 4: M2-M5 configuration : $(m, n) = (3, 3)$.

5 BPS equations with less supersymmetries

In the previous section, we have studied 1/2 BPS conditions that preserve 6 SUSY among 12 SUSY in the ABJM model. Once we consider more general configurations of M-theory branes, it is possible to partially break the 6 SUSY down up to 1 SUSY. In this section, we investigate BPS configurations with less than 6 supersymmetries. In the first half of the following subsections, we consider M2 and M5-branes that intersect with our fiducial M2-branes where the M2 and M5-branes have non-trivial angles with the orbifolded planes. Such a classification of M-branes in terms of angles are deeply investigated in [26]. In the latter half, we discuss more general configurations with multiple kinds of M2, M5-branes which correspond to more than one projection conditions.

5.1 M2-M2 intersections with angles

In this subsection, we consider M2-branes which intersect with our fiducial M2-branes at a point. The fiducial M2-branes are extending along $x^{0,1,2}$ -directions while the other M2-branes spans in the two-dimensional subspace of $\mathbb{C}^4/\mathbb{Z}_k$ which is transverse to the $x^{0,1,2}$ -directions.

Let v_1^I, v_2^I ($I = 3, 4, \dots, 10$) be two linearly independent vectors indicating a plane in \mathbb{R}^8 along which M2-branes are extending. The corresponding 1/2 BPS projection matrix in eleven dimensions is

$$\hat{\Gamma} = \frac{1}{2}(v_1^I v_2^J - v_2^I v_1^J) \Gamma_{0IJ}. \quad (5.1)$$

Here, the normalization of the vectors should be determined from the condition $\hat{\Gamma}^2 = \mathbf{1}$. This operator $\hat{\Gamma}$ is invariant up to normalization under $GL(2, \mathbb{R})$ transformations which mix the vectors v_1 and v_2 . Therefore, M2-branes are specified by points on the Grassmannian $G(2, \mathbb{R}^8)$, which is described by 8-by-2 matrices with the following equivalence relation

$$\begin{pmatrix} v_1^3 & v_2^3 \\ v_1^4 & v_2^4 \\ \vdots & \vdots \\ v_1^{10} & v_2^{10} \end{pmatrix} \sim \begin{pmatrix} v_1^3 & v_2^3 \\ v_1^4 & v_2^4 \\ \vdots & \vdots \\ v_1^{10} & v_2^{10} \end{pmatrix} g, \quad \forall g \in GL(2, \mathbb{R}). \quad (5.2)$$

However, instead of real 8-by-2 matrix, it is convenient to use the following complex 4-by-2 matrix to make the $SU(4)_R$ symmetry manifest

$$\Lambda_{M2} \equiv \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \\ u_1^3 & u_2^3 \\ u_1^4 & u_2^4 \end{pmatrix}, \quad (5.3)$$

where u_1^A and u_2^A are complex vectors corresponding to v_1 and v_2

$$u_1^A = v_1^{2A+1} + i v_1^{2A+2}, \quad u_2^A = v_2^{2A+1} + i v_2^{2A+2}. \quad (5.4)$$

The $SU(4)_R$ transformation acts on Λ_{M2} from the left and $GL(2, \mathbb{R})$ matrices acts from the right $\Lambda_{M2} \sim \Lambda_{M2} g$. Once the matrix Λ_{M2} is given the vectors v_1^I, v_2^I and the operator (5.1) can be uniquely determined. Without loss of generality, we can always fix Λ_{M2} in the following form by using $SU(4) \times U(1)$ and $GL(2, \mathbb{R})$ transformations

$$\Lambda_{M2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ i \sin \theta & 0 \\ \cos \theta & i \end{pmatrix}. \quad (5.5)$$

For this form of Λ_{M2} , the operator (5.1) is given by

$$\hat{\Gamma} = \hat{\Gamma}_0 (\sin \theta \hat{\Gamma}_8 + \cos \theta \hat{\Gamma}_9) \hat{\Gamma}_{10}. \quad (5.6)$$

Under the maps f and \tilde{f} , the operator $\hat{\Gamma}$ reduces to $\mathcal{A} = \gamma_0 \otimes \Xi$ and $\tilde{\mathcal{A}} = \gamma_0 \otimes \tilde{\Xi}$ with

$$\Xi = -g^T \text{diag} (i\sigma_2, i\sigma_2, i \cos \theta \sigma_2) g, \quad \tilde{\Xi} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\sin \theta & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin \theta \end{pmatrix} g, \quad (5.7)$$

where g is an $SO(6)$ matrix. For generic values of the angle parameter θ , there are 4 components of the spinor parameters ϵ satisfying

$$\mathcal{A} \epsilon = \epsilon, \quad \tilde{\mathcal{A}} \epsilon = 0. \quad (5.8)$$

The first condition reduces 12 SUSY down to 4 SUSY which is consistent with the second condition. As a result, 4 SUSY among 12 SUSY is preserved by the projection conditions (5.8). Therefore, M2-branes with generic values of the angle parameter θ are described by 1/3 BPS equations in the ABJM model⁴. The corresponding configuration is summarized in Table 5. On the other hand, the supersymmetry enhances to 6 SUSY (1/2 BPS) at $\theta = 0$ and $\theta = \pi$ since the conditions (4.26) are satisfied and the M2-branes do not extend across the line between two different orbifolded planes.

⁴ Note that the M2-branes themselves are orthogonal with each other and, as one can see in Table 5, θ parameterizes the angle between the second (not the fiducial) M2-branes and the hyperplanes on which the orbifold projection acts, see (4.2).

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M2	•								○ sin θ	○ cos θ	•

Table 5: 1/3 BPS configuration of intersecting M2-branes.

5.2 M2-M5 intersections with angles

Next, let us consider M5-branes extending along x^1 -direction and an arbitrary four-dimensional subspace in \mathbb{R}^8 spanned by four linearly independent vectors v_a^I ($a = 1, 2, 3, 4$). The corresponding operator in eleven dimensions is

$$\hat{\Gamma} \propto \frac{1}{4!} \epsilon^{abcd} v_a^I v_b^J v_c^K v_d^L \hat{\Gamma}_{01IJKL}. \quad (5.9)$$

The operators of this form are specified by points on the Grassmannian $G(4, \mathbb{R}^8)$. As in the previous case, let us define complex vectors $u_a^A \equiv v_a^{2A+1} + i v_a^{2A+2}$ ($a = 1, 2, 3, 4$) and the 4-by-4 complex matrix Λ_{M5} by

$$\Lambda_{M5} \equiv \begin{pmatrix} u_1^1 & u_2^1 & u_3^1 & u_4^1 \\ u_1^2 & u_2^2 & u_3^2 & u_4^2 \\ u_1^3 & u_2^3 & u_3^3 & u_4^3 \\ u_1^4 & u_2^4 & u_3^4 & u_4^4 \end{pmatrix}, \quad (5.10)$$

where $SU(4)$ transformations act from the left and $GL(4, \mathbb{R})$ transformations act from the right $\Lambda_{M5} \sim \Lambda_{M5} g$. By using the $SU(4)$ and $GL(4, \mathbb{R})$ transformations, the matrix Λ_{M5} can always be fixed as

$$\Lambda_{M5} = \begin{pmatrix} 1 & i \cos \theta_1 & 0 & 0 \\ 0 & \sin \theta_1 & 0 & 0 \\ 0 & 0 & 1 & i \cos \theta_2 \\ 0 & 0 & 0 & \sin \theta_2 \end{pmatrix}. \quad (5.11)$$

For this matrix Λ_{M5} , the operator $\hat{\Gamma}$ is mapped by f and \tilde{f} to $\mathcal{A} = \gamma_0 \otimes \Xi$ and $\tilde{\mathcal{A}} = \gamma_0 \otimes \tilde{\Xi}$ with

$$\Xi = g^T \text{diag} (1, 1, \cos(\theta_1 - \theta_2), \cos(\theta_1 + \theta_2), -1, -1) g, \quad (5.12)$$

$$\tilde{\Xi} = \begin{pmatrix} 0 & 0 & -\sin(\theta_1 - \theta_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin(\theta_1 + \theta_2) & 0 & 0 \end{pmatrix} g, \quad (5.13)$$

where g is an $SO(6)$ matrix. For generic values of the angle parameters θ_1 and θ_2 , the conditions $\mathcal{A}\epsilon = \epsilon$ and $\tilde{\mathcal{A}}\epsilon = 0$ are satisfied by 4 components of the spinor parameters ϵ . Namely, 4 SUSY

among 12 SUSY is preserved (1/3 BPS). See Table 6 for the configuration. If either of the angle parameters $\theta_1 \pm \theta_2$ becomes 0 or π , the supersymmetry enhances to 5 SUSY (5/12 BPS). If both of the angle parameters satisfy $\theta_1 \pm \theta_2 = 0$ or π , the preserved supersymmetry becomes 6 SUSY (1/2 BPS) and the corresponding configurations are those in Table 3 ($\theta_1 = 0, \theta_2 = 0$) or Table 4 ($\theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}$).

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M5	•	•		•	○ cos θ_1	○ sin θ_1		•	◇ cos θ_2	◇ sin θ_2	

Table 6: 1/3 BPS configuration of M2/M5-branes.

5.3 M2-M2 intersections with less supersymmetries

So far, we have considered only single sets of conditions $\mathcal{A}\epsilon = \epsilon$, $\tilde{\mathcal{A}}\epsilon = 0$ corresponding to a single type of BPS objects. If several sets of conditions are imposed on the spinor parameter ϵ , we can obtain BPS equations which admit multiple types of BPS objects. First, let us classify BPS configurations consisting of M2-branes only. Let $\Xi^{(I,J)}$ be 6-by-6 matrices corresponding to M2-branes extending along $x^{I,J}$ -directions by

$$\gamma_0 \otimes \Xi^{(I,J)} = P \hat{\Gamma}_{0IJ} P^\dagger. \quad (5.14)$$

We first study a condition that preserve 4 supercharges. To find such a condition, we consider two sets of M2-branes extending along $x^{5,6}$ and $x^{9,10}$ -directions. The corresponding projection matrices $\hat{\Gamma}_{056}, \hat{\Gamma}_{09(10)}$ are mapped to the one in ABJM model leading to the following conditions

$$(-\gamma_0 \otimes \Xi^{(5,6)})\epsilon = \epsilon, \quad (\gamma_0 \otimes \Xi^{(9,10)})\epsilon = \epsilon, \quad (5.15)$$

where $\Xi^{(5,6)}$ and $\Xi^{(9,10)}$ are given by

$$\Xi^{(5,6)} = -\text{diag}(-i\sigma_2, -i\sigma_2, i\sigma_2), \quad \Xi^{(9,10)} = -\text{diag}(i\sigma_2, i\sigma_2, i\sigma_2). \quad (5.16)$$

Clearly, the above conditions force the 8 SUSY among 12 SUSY to vanish resulting 4 remaining SUSY. The conditions (5.15) correspond to an anti-M2-branes extending along $x^{5,6}$ -directions and an M2-branes extending along $x^{9,10}$ -directions (Table 7)⁵. In addition, arbitrary M2-branes

⁵ Here the overline on the "M2" means that the corresponding eleven-dimensional projector has extra minus sign compared with the "M2" without overline which is defined through the projector of the form Γ_{0IJ} . We call this anti-M2-branes. In the following, we use the overline to denote this interpretation.

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
$\overline{\text{M2}}$	•					•	•				
M2	•									•	•

Table 7: 1/3 BPS intersecting M2-branes.

can be added without breaking further supersymmetry if the corresponding matrix Ξ takes the following form

$$\Xi = -\text{diag}(i\sigma_2, i\sigma_2, ic\sigma_2), \quad (5.17)$$

where c is an arbitrary real parameter. Let us see this fact in more detail. Consider M2-branes extending along a plane spanned by two vectors v_1 and v_2 , then we find that the corresponding BPS projection matrix $\hat{\Gamma} = v_1^I v_2^J \hat{\Gamma}_{0IJ}$ reduces to Ξ of the form (5.17) if the matrix Λ_{M2} defined in (5.3) is given by

$$\Lambda_{\text{M2}} = \begin{pmatrix} 0 & 0 \\ a & -ia \\ 0 & 0 \\ b & ib \end{pmatrix}, \quad (5.18)$$

where a and b are complex parameters satisfying $|a|^2 + |b|^2 = 1$ and $|b|^2 - |a|^2 = c$. This form of the matrix Λ_{M2} indicates a three-dimensional submanifold of the Grassmannian $G(2, \mathbb{R}^8)$ corresponding to a family of two-dimensional planes in \mathbb{R}^8 . Note that the condition $\tilde{\mathcal{A}}\epsilon = 0$ is automatically satisfied for the spinor parameter ϵ satisfying (5.15) since $\tilde{\mathcal{A}}$ takes the form

$$\tilde{\mathcal{A}} = 2\gamma_0 \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \text{Im}(ab) & \text{Re}(ab) \\ 0 & 0 & 0 & 0 & \text{Re}(ab) & -\text{Im}(ab) \end{pmatrix}. \quad (5.19)$$

Therefore, the 1/3 BPS equation of this type admits three parameter family of M2-branes.

The 1/6 BPS equations can be obtained by adding the following supersymmetry projection condition to (5.15)

$$(\gamma_0 \otimes \Xi^{(7,8)})\epsilon = \epsilon, \quad \Xi^{(7,8)} = -\text{diag}(i\sigma_2, -i\sigma_2, -i\sigma_2). \quad (5.20)$$

This condition corresponds to M2-branes extending along $x^{7,8}$ -directions. In this case, we can add arbitrary M2-branes whose matrix Ξ has the following form

$$\Xi = - \left(\begin{array}{c|c} i\sigma_2 & \\ \hline & *_{\text{4}} \end{array} \right), \quad (5.21)$$

where $*_4$ denotes arbitrary entries. For example, we can add anti-M2-branes in $x^{3,4}$ -directions since the corresponding projection condition takes the form

$$(-\gamma_0 \otimes \Xi^{(3,4)})\epsilon = \epsilon, \quad \Xi^{(3,4)} = -\text{diag}(-i\sigma_2, i\sigma_2, -i\sigma_2). \quad (5.22)$$

An example of the 1/6 BPS intersecting M2-branes is given in Table 8. In general, the BPS

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
$\overline{\text{M2}}$	•			•	•						
$\overline{\text{M2}}$	•					•	•				
M2	•							•	•		
M2	•									•	•

Table 8: 1/6 BPS intersecting M2-branes.

projection matrix $\hat{\Gamma} = v_{[1}^I v_{2]}^J \hat{\Gamma}_{0IJ}$ reduces to the matrix of the form (5.21) if the matrix Λ_{M2} is given by

$$\Lambda_{\text{M2}} = \left(\begin{array}{c|c} U_1 & \\ \hline & U_2 \end{array} \right) \begin{pmatrix} 0 & 0 \\ a & -ia \\ 0 & 0 \\ b & ib \end{pmatrix}, \quad (5.23)$$

where $|a|^2 + |b|^2 = 1$ and $U_1, U_2 \in SU(2)$. Therefore, compared with the 1/3 BPS equation, the 1/6 BPS equation of this type admits larger class of M2-branes rotated by $SU(2) \times SU(2) \subset SU(4)_R$. It is worthwhile to note that the M2-branes can be accompanied by KK-monopoles through the Hodge dualized projection conditions of the spinor parameters.

5.4 M2-M5 intersections with less supersymmetries

Let us next consider M5-branes which share the x^1 -direction with our fiducial M2-branes. The BPS configurations are classified by the chirality (eigenvalues of γ_2) of the preserved supercharges. In a BPS configuration with $\mathcal{N} = (m, n)$ preserved supercharges, the M5-branes are characterized by the condition $\gamma_2 \Xi_{ij} \epsilon_j = \epsilon_i$ with the symmetric matrix Ξ of the form

$$\Xi = \left(\begin{array}{c|c|c} \mathbf{1}_m & & \\ \hline & -\mathbf{1}_n & \\ \hline & & *_{{6-m-n}} \end{array} \right), \quad m + n \leq 6, \quad (5.24)$$

where $*_{6-m-n}$ is an arbitrary entry. Now, let us determine the matrix Λ_{M5} (defined in (5.10)) corresponding to Ξ of the form (5.24). Since arbitrary symmetric matrix Ξ can be diagonalized by $SO(6)_R$ transformations, it is sufficient to determine a matrix Λ_{M5} corresponding to the diagonal matrix of the form

$$\Xi = \text{diag}(\overbrace{1, \dots, 1}^m, \overbrace{-1, \dots, -1}^n, \overbrace{*, \dots, *}^{6-m-n}). \quad (5.25)$$

The generic forms of Λ_{M5} can be obtained by using $Spin(m) \times Spin(n) \times Spin(6-m-n)$ transformations corresponding to $SO(m) \times SO(n) \times SO(6-m-n)$ transformations which leave the form of (5.24) unchanged.

First, we note the fact that any 4-by-4 matrix Λ_{M5} can be fixed in the following form by using the $GL(4, \mathbb{R})$ and $SU(4)_R$ transformations

$$\Lambda_{M5} = \begin{pmatrix} \sin \theta_1 & -i \sin \theta_1 & 0 & 0 \\ 0 & 0 & \cos \theta_2 & i \cos \theta_2 \\ \cos \theta_1 & i \cos \theta_1 & 0 & 0 \\ 0 & 0 & \sin \theta_2 & -i \sin \theta_2 \end{pmatrix}. \quad (5.26)$$

For this matrix Λ_{M5} , the matrix $\hat{\Gamma}$ reduces to $\mathcal{A} = \gamma_2 \otimes \Xi$ with

$$\Xi = \text{diag} \left(1, 1, -1, -1, \cos[2(\theta_1 + \theta_2)], \cos[2(\theta_1 - \theta_2)] \right). \quad (5.27)$$

Therefore, the matrix Ξ for arbitrary M5-branes extending along a four-dimensional plane in $\mathbb{C}^4/\mathbb{Z}_k$ can be obtained from (5.27) by using $SO(6) \cong SU(4)_R$ rotations. If $SO(6)$ is restricted to $SO(5)$ generated by Σ_{ij} ($i, j = 2, \dots, 6$), the matrix Ξ takes the form

$$\Xi = \left(\begin{array}{c|c} 1 & \\ \hline & *_{\mathbf{5}} \end{array} \right). \quad (5.28)$$

For this form of the matrix, the BPS configurations preserve $\mathcal{N} = (m, n) = (1, 0)$ supercharges (1/12 BPS) and the corresponding matrix Λ_{M5} can be obtained from (5.26) by using $Spin(5) \cong USp(4)$ transformations defined by

$$U^T \Gamma_1 U = \Gamma_1, \quad U^\dagger U = \mathbf{1}_4. \quad (5.29)$$

Therefore, the 1/12 BPS equation preserving $\mathcal{N} = (1, 0)$ SUSY admits a family of M5-branes specified by

$$\Lambda_{M5} = U^\dagger \begin{pmatrix} \sin \theta_1 & -i \sin \theta_1 & 0 & 0 \\ 0 & 0 & \cos \theta_2 & i \cos \theta_2 \\ \cos \theta_1 & i \cos \theta_1 & 0 & 0 \\ 0 & 0 & \sin \theta_2 & -i \sin \theta_2 \end{pmatrix} U, \quad U \in USp(4). \quad (5.30)$$

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M5	•	•		•		•			•		•
$\overline{\text{M5}}$	•	•		•			•	•			•
$\overline{\text{M5}}$	•	•			•	•		•			•
$\overline{\text{M5}}$	•	•			•		•		•		•
M5	•	•		•	•					•	•
M5	•	•				•	•			•	•
$\overline{\text{M5}}$	•	•						•	•	•	•

Table 9: An example of $\mathcal{N} = (1, 0)$ BPS configuration. The Hodge-dual branes are omitted.

If we choose a specific matrix U and angles (θ_1, θ_2) , we find that $\mathcal{N} = (1, 0)$ SUSY admits a configuration given in Table 9.

In the above, we have used $Spin(5) \cong USp(4)$ to obtain the most generic form of the matrix Λ_{M5} which specify the M5-branes in $\mathcal{N} = (1, 0)$ configurations. If we restrict the transformation matrix U to $Spin(4) \subset Spin(5)$ defined by

$$U^T \Gamma_i U = \Gamma_i, \quad (i = 1, 2), \quad (5.31)$$

the matrix Ξ is rotated by $SO(4)$ generated by Σ_{ij} ($i, j = 3, \dots, 6$) and takes the form

$$\Xi = \left(\begin{array}{c|c} \mathbf{1}_2 & \\ \hline & *_{\mathbf{4}} \end{array} \right). \quad (5.32)$$

Therefore, the most general form of the vectors for M5-branes in $\mathcal{N} = (2, 0)$ configurations can be obtained from (5.26) by using $Spin(4) \cong SU(2) \times SU(2)$. An example of $\mathcal{N} = (2, 0)$ configuration is given in Table 10.

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M5	•	•		•	•					•	•
M5	•	•				•	•			•	•
$\overline{\text{M5}}$	•	•						•	•	•	•

Table 10: $\mathcal{N} = (2, 0)$ BPS configuration. The Hodge-dual branes are omitted.

To specify the form of the matrix Λ_{M5} corresponding to $\mathcal{N} = (1, 1)$ configurations, we have to first determine a diagonal matrix of the form

$$\Xi = \text{diag}(1, -1, *, *, *, *). \quad (5.33)$$

This form of the matrix Ξ can be obtained from (5.27) by using $\exp(i\frac{\pi}{2}\Sigma_{13}) \in SO(6)$, which corresponds to the $SU(4)$ element given by

$$U' \equiv \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \mathbf{1}_2 & \mathbf{1}_2 \\ \hline -\mathbf{1}_2 & \mathbf{1}_2 \end{array} \right). \quad (5.34)$$

Therefore, the most general form of the vectors for M5-branes in $\mathcal{N} = (1, 1)$ configurations can be obtained from (5.26) by acting U' and then using $Spin(4) \cong SU(2) \times SU(2)$ transformations. An example of $\mathcal{N} = (1, 1)$ configuration is given in Table 11.

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M5	•	•		•		•			•		•
$\overline{\text{M5}}$	•	•		•			•	•			•
$\overline{\text{M5}}$	•	•			•	•		•			•
$\overline{\text{M5}}$	•	•			•		•		•		•

Table 11: $\mathcal{N} = (1, 1)$ BPS configuration. The Hodge-dual branes are omitted.

By similar discussions, we can determine the matrix Λ_{M5} for $\mathcal{N} = (m, n)$ configurations. Note that the BPS equations considered here admits not only M5-branes extending along a four-dimensional plane in $\mathbb{C}^4/\mathbb{Z}_k$ but also other types of BPS objects. For example, the BPS equations with $\mathcal{N} = (m, 0)$ admits the wave-type and its Hodge dual M9-brane configurations. The complete list of BPS equations for $\mathcal{N} = (m, n)$ are given in Appendix A.

5.5 M2-branes ending on intersecting M5-branes

In the previous subsection, we have analyzed M2-M5 configurations in which all branes share one common direction (x^1 -direction). Here, let us consider BPS configurations of M2-branes ending on M5-branes extending along both $x^{1,2}$ -directions. For such configurations, the condition for the preserved supercharges are given by

$$\gamma_2 \Xi_{ij} \epsilon_j = \epsilon_i, \quad \gamma_1 \Xi'_{ij} \epsilon_j = \epsilon_i, \quad (5.35)$$

where Ξ and Ξ' are 6-by-6 symmetric real matrix. The most supercharges are preserved when $[\gamma_2 \Xi, \gamma_1 \Xi'] = 0$ and $\Xi^2 = \Xi'^2 = \mathbf{1}_6$. Up to $SU(4)_R$ transformations, the matrices Ξ and Ξ' satisfying these conditions are given by

$$\Xi = \text{diag}(\sigma_3, \sigma_3, \sigma_3), \quad \Xi' = \text{diag}(\sigma_1, \sigma_1, \sigma_1). \quad (5.36)$$

For these matrices, the condition (5.35) is satisfied by three components of the spinor parameter ϵ . Therefore the projections with these matrices are 1/4 BPS conditions. The corresponding matrices Λ_{M5} and Λ'_{M5} for the matrices Ξ and Ξ' are given by

$$\Lambda_{\text{M5}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Lambda'_{\text{M5}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}. \quad (5.37)$$

In this BPS configuration, the M5-branes have three-dimensional common world-volume ($x^{3,6,8}$ -directions). Obviously, we can also add objects specified by

$$\gamma_0 \Xi''_{ij} \epsilon_j = \epsilon_i, \quad \Xi'' \equiv \Xi \Xi' = \text{diag}(i\sigma_2, i\sigma_2, i\sigma_2). \quad (5.38)$$

This implies that M2-branes which intersect our fiducial M2-branes at a point can also be added to the 1/4 BPS configuration. An example of the 1/4 BPS configuration is given in Table 12. There are also BPS configurations with two preserved supercharges (1/6 BPS), for which the

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
$\overline{\text{M2}}$	•									•	•
M5	•	•		•			•		•		•
$\overline{\text{M5}}$	•		•		•	•		•			•

Table 12: 1/4 BPS configuration. The Hodge-dual branes are omitted.

matrices Ξ , Ξ' and Ξ'' are given by

$$\Xi = \text{diag}(\sigma_3, \sigma_3, *_2), \quad \Xi' = \text{diag}(\sigma_1, \sigma_1, *_2), \quad \Xi'' = \text{diag}(i\sigma_2, i\sigma_2, *_2). \quad (5.39)$$

These matrices correspond to M5-branes and M2-branes and the matrices Λ_{M5} and Λ_{M2} for them can be determined in a similar way as in the previous sections. The most generic BPS configurations in this class preserve one supercharge (1/12 BPS) specified by

$$\Xi = \left(\begin{array}{c|c} \sigma_3 & \\ \hline & *_4 \end{array} \right), \quad \Xi' = \left(\begin{array}{c|c} \sigma_1 & \\ \hline & *_4 \end{array} \right). \quad (5.40)$$

An example of 1/6 and 1/12 BPS configuration is given in Table 13 and 14. The BPS equations for these matrices are summarized in Appendix A.

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M2	•					•	•				
$\overline{\text{M2}}$	•									•	•
M5	•	•		•			•		•		•
$\overline{\text{M5}}$	•	•			•		•	•			•
M5	•		•	•		•			•		•
$\overline{\text{M5}}$	•		•		•	•		•			•

Table 13: 1/6 BPS configuration. The Hodge-dual branes are omitted.

6 Reduction to $\mathcal{N} = 8$ super Yang-Mills theory

In this section, we study the ten-dimensional interpretation of our BPS conditions by reducing the ABJM model to the multiple D2-brane effective theory, namely $(2+1)$ -dimensional $\mathcal{N} = 8$ super Yang-Mills theory. It is useful to analyze the reduction of the BPS conditions in the ABJM model to those in the super Yang-Mills theory since much more details of brane configurations are known in ten-dimensional string theory. The multiple M2-brane effective action is reduced to that of the D2-branes in type IIA string theory once one of the transverse direction of the M2-brane world-volume is compactified on S^1 . In the massless ABJM case, this can be achieved through the novel Higgs mechanism [14, 27]. We briefly summarize this procedure in the following.

First, we assume that the scalar fields Y^A in ABJM model develop a diagonal VEV $v^A \mathbf{1}_N$ and then the gauge symmetry $U(N) \times U(N)$ is broken down to $U(N)_{\text{diag}}$. Let us consider fluctuations around the VEV

$$Y^A = v^A \mathbf{1}_N + \frac{1}{2|v^A|} (X^{2A+1} + iX^{2A+2}), \quad (6.1)$$

where X^{2A+1}, X^{2A+2} are hermitian matrices in the adjoint representation of $U(N)_{\text{diag}}$. Next, we decompose the gauge fields A_μ, \hat{A}_μ as

$$A_\mu = a_\mu + b_\mu, \quad \hat{A}_\mu = a_\mu - b_\mu. \quad (6.2)$$

The gauge field a_μ becomes dynamical having its kinetic term while b_μ becomes an auxiliary field. The algebraic equation of motion for b_μ is solved as

$$b_\mu = -\frac{1}{4|v^A|^2} \left[\mathcal{D}_\mu X^\phi - \frac{1}{2} \epsilon_{\mu\nu\rho} f^{\nu\rho} \right] + \mathcal{O}(|v^A|^{-3}), \quad (6.3)$$

where \mathcal{D}_μ is the covariant derivative with a_μ , $f_{\mu\nu}$ is the field strength of a_μ and X^ϕ is defined

	0	1	2	3	4	5	6	7	8	9	10
M2	•	•	•								
M2	•			•	•						
M2	•					•	•				
$\overline{\text{M2}}$	•							•	•		
$\overline{\text{M2}}$	•									•	•
M5	•	•		•		•		•			•
M5	•	•			•	•			•		•
M5	•	•		•			•		•		•
$\overline{\text{M5}}$	•	•			•		•	•			•
M5	•		•	•		•			•		•
$\overline{\text{M5}}$	•		•		•	•		•			•
$\overline{\text{M5}}$	•		•	•			•	•			•
$\overline{\text{M5}}$	•		•		•		•		•		•

Table 14: 1/12 BPS configuration. The Hodge-dual branes are omitted.

by

$$X^\phi \equiv \frac{v_A^\dagger (X^{2A+1} + iX^{2A+2}) - v^A (X^{2A+1} - iX^{2A+2})}{2i|v^A|} = \frac{1}{|\vec{v}|} (J\vec{v}) \cdot \vec{X}, \quad (6.4)$$

where $J = i(\hat{\Sigma}_{34} + \hat{\Sigma}_{56} + \hat{\Sigma}_{78} + \hat{\Sigma}_{9,10})$ is a complex structure on $\mathbb{C}^4/\mathbb{Z}_k$. Finally, by taking the limit $v \rightarrow \infty, k \rightarrow \infty$ with fixed $1/g_{\text{YM}}^2 = \frac{k}{8\pi|v^A|^2}$, the procedure substantially leads to the compactification of the M-theory circle with finite gauge coupling. Substituting the solution (6.3) into the action of the ABJM model and taking the limit, we obtain the action of the $\mathcal{N} = 8$ super Yang-Mills theory with gauge group $U(N)$ and the gauge coupling g_{YM} . In the following, we set the VEV in a specific direction $A = 4$ for simplicity,

$$v^A = |v|\delta^{A4}, \quad 0 < v \in \mathbb{R}. \quad (6.5)$$

In this case we have $X^\phi = X^{10}$. This means the x^{10} -direction is the M-theory circle. The bosonic part of the action becomes

$$S_{\text{bosonic}} = \frac{1}{g_{\text{YM}}^2} \int d^3x \text{Tr} \left[-\frac{1}{2} f_{\mu\nu} f^{\mu\nu} - \mathcal{D}_\mu X^I \mathcal{D}^\mu X^I - \frac{1}{2} [X^I, X^J]^2 \right]. \quad (6.6)$$

The supersymmetric variation of the gaugino in three-dimensional $\mathcal{N} = 8$ super Yang-Mills theory is given by

$$\delta\psi = \partial_\mu X_I \Gamma^\mu \Gamma^I \xi + \frac{1}{2} f_{\mu\nu} \Gamma^{\mu\nu} \Gamma^{10} \xi + \frac{i}{2} [X_I, X_J] \Gamma^{IJ} \Gamma^{10} \xi, \quad (6.7)$$

where Γ^μ ($\mu = 0, 1, 2$) and Γ^I ($I = 3, \dots, 9$) are ten-dimensional gamma matrices. The supersymmetry parameter ξ is an $SO(9, 1)$ Majorana spinor that satisfies $\xi = \Gamma^{012}\xi$.

The reduction to ten-dimensions can be applied also to the BPS equations in the ABJM model. The reduced conditions are BPS equations in the effective theory of N coincident D2-branes in type IIA string theory. Let us see this procedure especially focusing on the 1/2 BPS, the most restricted, and 1/12 BPS, the most generic, equations discussed in sections 3, 4 and 5.

6.1 Reduction of 1/2 BPS equations

We start from the 1/2 BPS equations discussed in section 3. There are basically two types of 1/2 BPS equations corresponding to $\mathcal{A} = \gamma_0 \otimes B$ and $\mathcal{A} = \gamma_2 \otimes C^{(m,n)}$. The former condition gives the BPS equations (3.12), (3.13) supplemented by the Gauss' law (3.14), (3.14). Solutions to these equations are given essentially as point-like object in the world-volume of M2-branes. The latter corresponds to the BPS equations (3.16), (3.17) with the Gauss' law having fuzzy funnel type solutions. In the following, we discuss the reduction of these BPS conditions separately. As we will see, all the 1/2 BPS equations in the ABJM model reduce to 1/2 BPS equations or the vacuum condition in D2-brane world-volume theory.

6.1.1 M2-M2 to D2-F1

Let us consider M2-branes extending along $x^{9,10}$ -directions which intersect with our fiducial M2-branes at a point. The projection operator for such configuration is $\mathcal{A} = \gamma_0 \otimes B$ where B is given by (3.7). If we compactify the x^{10} -direction in eleven dimensions, the first M2-branes are reduced to the type IIA fundamental strings (F1-string) while our fiducial M2-branes become N coincident D2-branes. Once we apply the reduction procedure to the corresponding BPS equation, we obtain the following equations,

$$f_{12} = -i[X^3, X^4] = -i[X^5, X^6] = -i[X^7, X^8], \quad (6.8)$$

$$0 = i[X^I, X^J], \quad (I, J) \neq (3, 4), (5, 6), (7, 8), (I, 9), \quad (6.9)$$

$$\mathcal{D}_0 X^I = -i[X^9, X^I], \quad \mathcal{D}_1 X^I = \mathcal{D}_2 X^I = 0 \quad (I = 3, \dots, 8), \quad (6.10)$$

$$\mathcal{D}_0 X^9 = 0, \quad \mathcal{D}_1 X^9 = f_{01}, \quad \mathcal{D}_2 X^9 = f_{02}. \quad (6.11)$$

On the other hand, the Gauss' law constraint reduces to the following equation

$$\mathcal{D}^\mu f_{\mu\nu} - i[X^I, \mathcal{D}_\nu X^I] = 0. \quad (6.12)$$

Using the SUSY variation (6.7), we find that the configuration specified by the reduced BPS equations and the Gauss' law preserve 6 SUSY among 16 SUSY characterized by

$$\Gamma^{09}\Gamma^{10}\xi = -\xi, \quad (\Gamma^{12} - \Gamma^{34} - \Gamma^{56} - \Gamma^{78})\xi = 0. \quad (6.13)$$

Using the conditions (6.8), (6.9), one can show that all X^I except for X^9 commute with each other $[X^I, X^J] = 0$. Then from (6.8), we find that it is possible to choose a gauge $a_1 = a_2 = 0$. The condition (6.10) with a gauge $a_0 = -X^9$ implies

$$\partial_0 X^I = 0 \implies X^I = \text{diag}(\text{const.}, \dots, \text{const.}), \quad (I = 3, \dots, 8). \quad (6.14)$$

In this case, solutions to the equations preserve at least 8 SUSY (1/2 BPS) determined by

$$\Gamma^{09}\Gamma^{10}\xi = -\xi. \quad (6.15)$$

This is nothing but the projection condition for F1-string extending along x^9 -direction [28]. Then the solutions are given by

$$a_0 = -X^9, \quad a_1 = a_2 = 0, \quad (\partial_1^2 + \partial_2^2)X^9 - [X^I, [X^I, X^9]] = 0. \quad (6.16)$$

Since X^I are all diagonal, it is easy to find a solution of this equation. The diagonal harmonic solution of X^9 determined by (6.16) is known as B1ons [29, 30] representing F1-strings ending on the D2-branes. On the other hand, the off-diagonal parts of X^9 have non-trivial solutions, giving more generic configurations of F1, such as strings stretched between various D2-branes.

It is worthwhile to note that as we have mentioned in section 4, it is possible that KK-monopoles and the M2-branes exist at the same time. We therefore expect that the reduced BPS equations (6.8) - (6.11) accommodates D6-branes in addition to the F1-strings. However there seems to be no solutions corresponding to the D6-branes since all the directions except X^9 are constant and do not show the D6-brane behavior. We discuss this issue in section 7.

6.1.2 M2-M2 to D2-D2

Next, let us consider the reduction of M2-branes extending along $x^{7,8}$ -directions. Since the directions are different from the one in the M-theory circle, the reduced objects should be D2-branes. The corresponding projection operator is

$$\Xi = \text{diag}(i\sigma_2, -i\sigma_2, -i\sigma_2). \quad (6.17)$$

It is straightforward to perform the reduction procedure and the result is the Hitchin equations given by

$$f_{12} - i[X^7, X^8] = 0, \quad (\mathcal{D}_1 + i\mathcal{D}_2)(X^7 - iX^8) = 0, \quad (6.18)$$

$$\begin{aligned} i[X^I, X^J] &= 0, & \mathcal{D}_1 X^I &= \mathcal{D}_2 X^I = 0, & (I \neq 7, 8), \\ f_{01} = f_{02} &= 0, & \mathcal{D}_0 X^I &= 0, & (I = 3, \dots, 9). \end{aligned} \quad (6.19)$$

These solutions preserve 8 SUSY among 16 SUSY, hence the condition corresponds to 1/2 BPS configuration. The projection condition is

$$\Gamma^{078}\xi = -\xi. \quad (6.20)$$

This is just the condition for D2-branes extending along $x^{0,7,8}$ -directions. Therefore the reduced condition implies D2-branes which intersect with the fiducial D2-branes.

6.1.3 $\mathcal{N} = (6, 0)$ M2-wave to D2-wave

Let us consider the $\mathcal{N} = (6, 0)$ BPS equations specified by the matrix

$$\Xi = \text{diag}(1, 1, 1, 1, 1, 1). \quad (6.21)$$

This has been discussed in section 4. The corresponding object is an M-wave extending along $x^{0,1}$ -directions shared with the fiducial M2-branes. After the reduction, the object becomes the type IIA wave in ten dimensions. The BPS equation reduces to

$$f_{02} - f_{12} = 0, \quad (\mathcal{D}_0 - \mathcal{D}_1)X^I = 0, \quad (6.22)$$

$$\begin{aligned} \mathcal{D}_2 X^9 &= i[X^3, X^4] = i[X^5, X^6] = i[X^7, X^8], \\ [X^I, X^J] &= 0, \quad (I, J) \neq (3, 4), (5, 6), (7, 8), \\ f_{01} &= 0, \quad \mathcal{D}_2 X^I = 0, \quad (I = 3, \dots, 8). \end{aligned} \quad (6.23)$$

As we have noticed, it is possible to show that there is no non-trivial (non-diagonal constant) solution that satisfies $[X^I, X^J] \neq 0$. Therefore preserved supersymmetry is actually 8 SUSY determined by

$$\Gamma^{01}\xi = \xi, \quad (6.24)$$

which is $\mathcal{N} = (8, 0)$ SUSY in (1+1)-dimensional theory ⁶.

6.1.4 $\mathcal{N} = (5, 1)$ condition to vacuum

Next, we consider the $\mathcal{N} = (5, 1)$ BPS equations. In general, the corresponding objects in M-theory is M5-branes extending along the calibrated submanifold in $\mathbb{C}^4/\mathbb{Z}_k$ specified by the 4-form (4.44). We can show that these BPS equations reduce to vacuum conditions in super Yang-Mills theory. This would mean that the corresponding objects in ten dimensions do not exist or it is non-BPS configuration.

⁶This (1 + 1) dimensional directions are common world-volume shared by both the waves (or other BPS objects) and the fiducial D2-branes. The statement " $\mathcal{N} = (m, n)$ " should be understood as the number of (anti)chiral supercharges that are preserved by the effective theory living in the common world-volume.

6.1.5 $\mathcal{N} = (4, 2)$ M2-M5 to D2-D4

Let us consider $\mathcal{N} = (4, 2)$ BPS condition representing M5-branes that extend along $x^{0,2,5,6,7,10}$ -directions. The corresponding projection operator is

$$\Xi = (1, 1, 1, 1, -1, -1). \quad (6.25)$$

Once we compactify x^{10} -directions, it correspond to direct dimensional reduction of the M5-branes leaving D4-branes in ten dimensional space-time. The reduced BPS equations are the Nahm equations

$$\mathcal{D}_2 X^5 = i[X^6, X^9], \quad \mathcal{D}_2 X^6 = i[X^9, X^5], \quad \mathcal{D}_2 X^9 = i[X^5, X^6], \quad (6.26)$$

$$\begin{aligned} f_{01} = f_{02} = f_{12} = \mathcal{D}_0 X^I = \mathcal{D}_1 X^I = 0, \\ \mathcal{D}_2 X^I = i[X^I, X^J] = 0, \quad (I = 3, 4, 7, 8, \quad J = 3, \dots, 9). \end{aligned} \quad (6.27)$$

The solutions preserve 8 SUSY determined by

$$\Gamma^{2569} \Gamma^{10} \xi = \xi \quad \leftrightarrow \quad \Gamma^{01569} \Gamma^{10} \xi = \xi. \quad (6.28)$$

This is nothing but the projection condition of D4-branes extending along $x^{0,1,5,6,9}$ -directions. The preserved SUSY is $\mathcal{N} = (4, 4)$ in $(1+1)$ -dimensions.

6.1.6 $\mathcal{N} = (3, 3)$ M2-M5 to D2-D4

Consider $\mathcal{N} = (3, 3)$ BPS condition that corresponds to M5-branes extending along $x^{0,2,5,6,9,10}$ -directions for which the projection operator is given by

$$\Xi = \text{diag}(-1, 1, 1, -1, -1, 1). \quad (6.29)$$

The BPS equations are reduced to the following Nahm equations

$$\mathcal{D}_2 X^4 = i[X^6, X^8], \quad \mathcal{D}_2 X^6 = i[X^8, X^4], \quad \mathcal{D}_2 X^8 = i[X^4, X^6], \quad (6.30)$$

$$\begin{aligned} f_{01} = f_{02} = f_{12} = \mathcal{D}_0 X^I = \mathcal{D}_1 X^I = 0, \\ \mathcal{D}_2 X^I = i[X^I, X^J], \quad (I = 3, 5, 7, 9, \quad J = 3, \dots, 9). \end{aligned} \quad (6.31)$$

The solution should represent D4-branes in ten dimensions. Actually, the preserved supersymmetry is characterized by

$$\Gamma^{2468} \Gamma^{10} \xi = \xi \quad \leftrightarrow \quad \Gamma^{01468} \Gamma^{10} \xi = \xi, \quad (6.32)$$

and represents D4-branes in $x^{0,1,4,6,8}$ -directions. The projection condition keeps $\mathcal{N} = (4, 4)$ SUSY in $(1+1)$ -dimensions.

6.2 Reduction of 1/12 BPS equations

So far, we have studied the reduction of the 1/2 BPS configurations in the ABJM model. There are several ways to find BPS equations that keep less than 6 SUSY. We have shown in section 5 that if some of intersecting branes in the 1/2 BPS configurations have non-trivial intersecting angles with orbifolded planes, the preserved 6 SUSY is further broken. In addition to this mechanism, once one imposes several kind of projection conditions or restricting the form of the projection matrices Ξ , the remaining supersymmetries can be reduced down to 1 SUSY among 12 SUSY. In the following subsections, we consider 1/12 BPS conditions derived in this way and see what kind of structures are obtained after the dimensional reduction to ten dimensions. We will see that the 1/12 BPS equations in the ABJM model reduce to 1/16 BPS equations in $\mathcal{N} = 8$ super Yang-Mills theory.

6.2.1 Intersecting M5-branes

Let us consider $\mathcal{N} = (1, 0)$ BPS equations with the projection operator

$$\Xi = \text{diag}(1, 0, 0, 0, 0, 0). \quad (6.33)$$

This corresponds to the fiducial M2-branes ending on several (anti) M5-branes together with M-waves. Performing the reduction, we obtain the following reduced equations

$$f_{02} = f_{12}, \quad f_{01} = 0 \quad (\mathcal{D}_0 - \mathcal{D}_1)X^I = 0, \quad \mathcal{D}_2 X^I = ig_{IJK}[X^J, X^K], \quad (6.34)$$

where the anti-symmetric structure constants g_{IJK} are defined by

$$g_{349} = g_{358} = -g_{367} = -g_{457} = -g_{468} = g_{569} = -g_{789} = 1/2. \quad (6.35)$$

This configuration preserves 1 SUSY, hence it is 1/16 BPS condition determined by

$$\Gamma^{2358}\Gamma^{10}\xi = \xi, \quad \Gamma^{2457}\Gamma^{10}\xi = -\xi, \quad \Gamma^{2367}\Gamma^{10}\xi = -\xi, \quad \Gamma^{01}\xi = \xi. \quad (6.36)$$

These conditions imply co-existence of D4-branes with type IIA-waves.

6.2.2 Intersecting M2-M5-branes

Let us consider BPS equations specified by

$$\Xi = \left(\begin{array}{c|c} \sigma_3 & \\ \hline & *_{\text{4}} \end{array} \right), \quad \Xi' = \left(\begin{array}{c|c} \sigma_1 & \\ \hline & *_{\text{4}} \end{array} \right). \quad (6.37)$$

The reduced equations are given by

$$f_{01} = \mathcal{D}_1 X^9, \quad f_{02} = \mathcal{D}_2 X^9, \quad f_{12} = -[Z^{\hat{A}}, Z_{\hat{A}}^{\dagger}] - [W, W^{\dagger}], \quad (6.38)$$

$$\mathcal{D}_0 Z^{\hat{A}} = i[Z^{\hat{A}}, X^9], \quad \mathcal{D}_0 W = i[W, X^9], \quad (6.39)$$

$$\mathcal{D}_{\bar{z}} Z^{\hat{A}} = \frac{i}{2} \epsilon^{\hat{A}\hat{B}} [Z_{\hat{B}}^{\dagger}, W^{\dagger}], \quad \mathcal{D}_{\bar{z}} W = \frac{i}{4} \epsilon^{\hat{A}\hat{B}} [Z_{\hat{A}}^{\dagger}, Z_{\hat{B}}^{\dagger}], \quad (6.40)$$

where we have defined $z = x^1 + ix^2$, $\mathcal{D}_z = \frac{1}{2}(\mathcal{D}_1 - i\mathcal{D}_2)$ and

$$Z^1 = X^3 + iX^4, \quad Z^2 = X^5 + iX^6, \quad W = X^7 - iX^8. \quad (6.41)$$

This configuration preserves 1 SUSY. Therefore it is 1/16 BPS condition specified by

$$\Gamma^{1357}\Gamma^{10}\xi = \xi, \quad \Gamma^{2358}\Gamma^{10}\xi = \xi, \quad \Gamma^{1458}\Gamma^{10}\xi = \xi, \quad \Gamma^{1368}\Gamma^{10}\xi = \xi, \quad (6.42)$$

These conditions imply the existence of the multiple kinds of D2 and D4-branes.

7 Conclusions and discussions

In this paper, we have classified BPS equations in the ABJM model derived from the vanishing conditions of $\mathcal{N} = 6$ supersymmetry transformation of the fermions. The BPS equations are characterized in terms of the unbroken supercharges specified by the projection conditions for the spinor parameters of the supersymmetry transformation. For the 1/2 BPS case, we have found the two types of the projection conditions (3.9) and (3.10), which correspond to co-dimension two and one objects, respectively.

The analysis of the projection conditions for the supersymmetry parameter provides not only classifications of BPS equations in the ABJM model, but also an insight into the BPS objects extending in eleven-dimensional space-time. We have discussed the projection conditions for eleven-dimensional spinor parameters that are consistent with the existence of our fiducial M2-branes in $\mathbb{C}^4/\mathbb{Z}_k$ orbifold. Those projection conditions give the information on how BPS objects extend in eleven-dimensional space-time. We have made the mapping (4.24) between the projection matrices in eleven dimensions and those in the ABJM model, and found the eleven-dimensional interpretation of the BPS objects in the ABJM model.

Starting from the 1/2 BPS conditions, we have also obtained $n/12$ ($n = 1, \dots, 5$) BPS conditions. A careful investigation of the SUSY breaking pattern allows us to establish the correspondence between each BPS equations and the possible brane configurations in M-theory. Those include M2, M5-branes which have non-trivial angles with the orbifolded planes and several bunches of M2, M5-branes filling subspaces in the internal space $\mathbb{C}^4/\mathbb{Z}_k$. We have also shown that the existence of various M-theoretical objects, such as M-waves, KK-monopoles and M9-branes, is consistent with the BPS conditions. These results are summarized in Table 15.

To see the dimensional reduction of the BPS equations is another consistency check. By taking the reduction limit via the novel Higgs mechanism the BPS equations in the ABJM

model reduce to those in the three-dimensional $\mathcal{N} = 8$ super Yang-Mills theory in a consistent way with the reduction from M-theory to Type IIA string theory. Our results reveal strong evidences that the ABJM model correctly captures dynamics of multiple M2-branes.

It is important to bear in mind that we discussed here the relation between the BPS equations in the ABJM model and the BPS objects in M-theory only in the aspects of the SUSY conditions. Of course the classification of the projection condition for SUSY parameters is not sufficient to conclude the correspondence between the BPS equations and the M-theoretical objects since the derived equations may have no non-trivial solutions. This is a conceivable argument since, as we have claimed in the previous section, there seem to be no non-trivial solutions in ten-dimensional BPS equations that correspond to D6, D8 and NS5-branes. This fact indicates that some of BPS equations in ABJM model have only trivial solutions. To complete the analysis, we should confirm the existence of solutions of the BPS equations, and examine whether the solutions have appropriate property as M2-brane, M5-brane, and so on. We have found a number of non-trivial solutions for the BPS equations derived in this paper. The detail discussions and explicit forms of these solutions will be found in the forthcoming paper [31].

\mathcal{N}	Residual symmetry	Intersecting branes
6	$SU(3) \times U(1)^2$	M2, KK-monopoles
6	$SU(4)$	M2, M9, M-waves
6	$SU(2) \times SU(2) \times U(1)^2$	M5
5	$SU(2) \times SU(2)$	M5 with angles
4	$U(1)^3$	M5 with angles
4	$SU(2) \times U(1)^2$	M2, KK-monopoles
4	$SU(2) \times U(1)^2$	M2 with angles
3	$SO(3)$	M2 ending on M5
2	$SU(2) \times SU(2) \times U(1)^2$	M2, KK-monopoles
2	$U(1) \times U(1)$	M2 ending on M5
1	$SU(2) \times SU(2)$	M2 ending on M5, M9, M-waves
$n + m$	$Spin(n) \times Spin(m) \times Spin(6 - n - m)$	M5, M9, M-waves

Table 15: Classification of the BPS equations in the number of preserved supercharges, the symmetry of BPS equations and the corresponding M-theoretical objects. \mathcal{N} is the number of preserved supercharges.

Acknowledgments

K. I. gratefully acknowledges the financial support from the Global Center of Excellence Program by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan through the “Nanoscience and Quantum Physics” Project of the Tokyo Institute of Technology, and support from the Iwanami Fujukai Foundation. This work is also supported by the Grant-in-Aid for the Global Center of Excellence Program of the Kyoto University “The Next Generation of Physics, Spun from Universality and Emergence” from MEXT. The work of K. I and S. S is supported by the Japan Society for the Promotion of Science (JSPS) Research Fellowship.

A BPS equations

In this appendix, we summarize the BPS equations discussed in this paper. In order to write down the BPS equations, it is convenient to define β_A^{BC} by

$$\beta_A^{BC} \equiv Y^B Y_A^\dagger Y^C - Y^C Y_A^\dagger Y^B. \quad (\text{A.1})$$

As we have mentioned in section 3, all the BPS conditions in this Appendix are supplemented by the Gauss’ law constraints (3.14), (3.14). This is the necessary condition that solutions of the BPS equations satisfy the equations of motion.

A.1 M2-branes

Let us consider the BPS configurations of intersecting M2-branes preserving the supercharges determined by the following condition

$$\gamma_0 \Xi_{ij} \epsilon_j = \epsilon_i, \quad (\text{A.2})$$

where Ξ is a 6-by-6 real anti-symmetric matrix satisfying $\Xi^2 = -\mathbf{1}_6$. The BPS equations given below preserve $2n$ ($n = 1, 2, 3$) supercharges and specified by the following matrices

$$\Xi = \begin{cases} \text{diag}(i\sigma_2, *_2, *_2) & \text{for } n = 1 \\ \text{diag}(i\sigma_2, i\sigma_2, *_2) & \text{for } n = 2 \\ \text{diag}(i\sigma_2, i\sigma_2, i\sigma_2) & \text{for } n = 3 \end{cases}. \quad (\text{A.3})$$

The corresponding BPS equations take the following form

$$-D_0 Y^B (\Gamma_j)_{BA} \Xi_{ji} + \Upsilon_A^{BC} (\Gamma_i)_{BC} = 0, \quad D_1 Y^B (\Gamma_i)_{BA} + D_2 Y^B (\Gamma_j)_{BA} \Xi_{ji} = 0, \quad (\text{A.4})$$

where $1 \leq i, j \leq 2n$. The BPS configurations saturate the BPS bound for the energy

$$E \geq \frac{k}{4\pi n} \int d^2 x \epsilon^{MN} \partial_M \text{Tr} \left[Y_C^\dagger D_N Y^B \right] (\Gamma_i \Gamma_j^\dagger \Xi_{ji})_B^C, \quad (M, N = 1, 2), \quad (\text{A.5})$$

and if the BPS equations are satisfied, the energy becomes

$$E = \frac{k}{2\pi} \int d^2x \partial_M \text{Tr} \left[Y_A^\dagger D^M Y^A \right]. \quad (\text{A.6})$$

In the following, we present detail structures of each $n = 1, 2, 3$ BPS conditions.

- 1/6 BPS equations

For the 1/6 BPS equations of intersecting M2-branes, the anti-symmetric matrix Ξ is given by

$$\Xi = - \left(\begin{array}{c|c} i\sigma_2 & \\ \hline & *_{\text{4}} \end{array} \right). \quad (\text{A.7})$$

Since the form of the matrix Ξ is invariant under $SO(4) \times SO(2)$, the corresponding BPS equations have $SU(2) \times SU(2) \times U(1)$ symmetry for which Y^a ($a = 1, 2$) and $Y^{\dot{a}}$ ($\dot{a} = 3, 4$) are in $(\mathbf{2}, \mathbf{1})_1$ and $(\mathbf{1}, \mathbf{2})_{-1}$ respectively. The BPS equations are given by

$$D_0 Y^a = i(\beta_b^{ab} - \beta_{\dot{b}}^{ab}), \quad (D_1 - iD_2)Y^a = 0, \quad (\text{A.8})$$

$$D_0 Y^{\dot{a}} = i(\beta_b^{\dot{a}b} - \beta_{\dot{b}}^{\dot{a}b}), \quad (D_1 + iD_2)Y^{\dot{a}} = 0, \quad (\text{A.9})$$

$$\beta_a^{\dot{a}b} = \beta_{\dot{a}}^{ab} = 0. \quad (\text{A.10})$$

- 1/3 BPS equations

For the 1/3 BPS equations of intersecting M2-branes, the anti-symmetric matrix Ξ is given by

$$\Xi = \left(\begin{array}{c|c|c} i\sigma_2 & & \\ \hline & i\sigma_2 & \\ \hline & & *_{\text{2}} \end{array} \right). \quad (\text{A.11})$$

The corresponding BPS equations are symmetric under $SU(2) \times U(1)^2$ for which Y^α ($\alpha = 1, 3$), Y^2 , Y^4 are in $\mathbf{2}_{(1,1)}$, $\mathbf{1}_{(-2,0)}$, $\mathbf{1}_{(0,-2)}$, respectively. The BPS equations are given by

$$D_0 Y^\alpha = i(\beta_2^{\alpha 2} - \beta_4^{\alpha 4}), \quad D_1 Y^\alpha = D_2 Y^\alpha = 0, \quad (\text{A.12})$$

$$D_0 Y^2 = -i\beta_4^{24}, \quad (D_1 - iD_2)Y^2 = 0, \quad (\text{A.13})$$

$$D_0 Y^4 = i\beta_2^{42}, \quad (D_1 + iD_2)Y^4 = 0, \quad (\text{A.14})$$

$$\beta_\alpha^{\beta\gamma} = 0, \quad \beta_2^{4\alpha} = \beta_4^{2\alpha} = 0, \quad \beta_\alpha^{2\beta} = \frac{1}{2}\delta_\alpha^\beta \beta_\gamma^{2\gamma}, \quad \beta_\alpha^{4\beta} = \frac{1}{2}\delta_\alpha^\beta \beta_\gamma^{4\gamma}. \quad (\text{A.15})$$

- 1/2 BPS equations

For the 1/2 BPS equations of intersecting M2-branes, the anti-symmetric matrix Ξ is given by

$$\Xi = - \left(\begin{array}{c|c|c} i\sigma_2 & & \\ \hline & i\sigma_2 & \\ \hline & & i\sigma_2 \end{array} \right). \quad (\text{A.16})$$

The corresponding BPS equations have $SU(3) \times U(1)^2$ for which Y^i ($i = 1, 2, 3$) and Y^4 are in $\mathbf{3}_1$ and $\mathbf{1}_{-3}$, respectively. The BPS equations are given by

$$D_0 Y^i = -i\beta_4^{i4}, \quad D_1 Y^i = D_2 Y^i = 0, \quad (\text{A.17})$$

$$D_0 Y^4 = \frac{i}{3}\beta_i^{4i}, \quad (D_1 + iD_2)Y^4 = 0, \quad (\text{A.18})$$

$$\beta_i^{jk} = \beta_4^{jk} = 0, \quad \beta_i^{4j} = \frac{1}{3}\delta_i^j \beta_k^{4k}. \quad (\text{A.19})$$

A.2 M5-branes

We summarize here the complete list of BPS equations for M5-branes in order of increasing unbroken supercharges. The unbroken supercharges are specified by

$$\gamma_2 \Xi_{ij} \epsilon_j = \epsilon_i, \quad (\text{A.20})$$

where Ξ is a 6-by-6 real symmetric matrix satisfying $\Xi^2 = \mathbf{1}_6$. For the BPS configurations with $\mathcal{N} = (m, n)$ supersymmetry, the symmetric matrix Ξ takes the form

$$\Xi = \left(\begin{array}{c|c|c} \mathbf{1}_m & & \\ \hline & -\mathbf{1}_n & \\ \hline & & *_{6-m-n} \end{array} \right). \quad (\text{A.21})$$

Since the form of the matrix is invariant under $SO(m) \times SO(n) \times SO(6-m-n)$ subgroup of $SO(6)_R$, the corresponding BPS equations have $Spin(m) \times Spin(n) \times Spin(6-m-n) \subset SU(4)_R$ symmetry. The $\mathcal{N} = (m, n)$ BPS equations are given by the following general formula;

$$D_0 Y^A = \pm D_1 Y^A, \quad (\text{if } n = 0 \text{ or } m = 0), \quad D_0 Y^A = D_1 Y^A = 0, \quad (\text{if } m, n \neq 0), \quad (\text{A.22})$$

$$D_2 Y^A = -\Upsilon_D^{BC} (\Gamma_i)_{BC} (\Gamma_i^\dagger)^{DA}, \quad (\text{no sum over } i, \quad i = 1, \dots, m), \quad (\text{A.23})$$

$$D_2 Y^A = \Upsilon_D^{BC} (\Gamma_j)_{BC} (\Gamma_j^\dagger)^{DA}, \quad (\text{no sum over } j, \quad j = m+1, \dots, m+n). \quad (\text{A.24})$$

The BPS configurations saturate the following BPS bound for the energy

$$E \geq \pm P_1 + \frac{k}{4\pi(m+n)} \int d^2x \partial_2 \text{Tr} \left[Y_A^\dagger \Upsilon_D^{BC} \right] \sum_{i,j=1}^{m+n} (\Gamma_i)_{BC} (\Gamma_j^\dagger)^{AD} \Xi_{ij}, \quad (\text{A.25})$$

where P_1 is the conserved momentum given by

$$P_1 \equiv \frac{k}{2\pi} \int d^2x \text{Tr} \left[D_0 Y_A^\dagger D_1 Y^A + D_1 Y_A^\dagger D_0 Y^A \right]. \quad (\text{A.26})$$

For any configurations satisfying the BPS equations, the energy is given by

$$E = \pm P_1 + \frac{k}{4\pi} \int d^2x \partial_2 \text{Tr} \left[Y_A^\dagger D_2 Y^A \right]. \quad (\text{A.27})$$

1/12 BPS equations

- $\mathcal{N} = (1, 0)$

The matrix Ξ for the 1/12 BPS configurations is invariant under $SO(5)$, so that the corresponding BPS equations have $Spin(5) \cong USp(4)$ symmetry defined by

$$U^T \Gamma_1 U = \Gamma_1, \quad U U^\dagger = \mathbf{1}_4. \quad (\text{A.28})$$

The scalar fields Y^A ($A = 1, 2, 3, 4$) are in $\mathbf{4}$ of $USp(4)$. The BPS equations are given by

$$D_2 Y^A = \beta_D^{AD} + (\Gamma_1^\dagger)^{AB} (\Gamma_1)_{CD} \beta_B^{CD}, \quad (D_0 - D_1) Y^A = 0. \quad (\text{A.29})$$

1/6 BPS equations

- $\mathcal{N} = (2, 0)$

For the BPS configurations with $\mathcal{N} = (2, 0)$ preserved supersymmetry, the symmetry of the corresponding BPS equations is $SU(2) \times U(1)$ for which Y^a ($a = 1, 2$) and $Y^{\dot{a}}$ ($\dot{a} = 3, 4$) are in $(\mathbf{2}, \mathbf{1})_1$ and $(\mathbf{1}, \mathbf{2})_{-1}$, respectively. The BPS equations are given by

$$D_2 Y^a = -\beta_b^{ab} + \beta_b^{a\dot{b}}, \quad (D_0 - D_1) Y^a = 0, \quad (\text{A.30})$$

$$D_2 Y^{\dot{a}} = -\beta_b^{\dot{a}b} + \beta_b^{\dot{a}\dot{b}}, \quad (D_0 - D_1) Y^{\dot{a}} = 0. \quad (\text{A.31})$$

$$\beta_b^{\dot{c}\dot{d}} = \beta_b^{cd} = 0. \quad (\text{A.32})$$

- $\mathcal{N} = (1, 1)$

For the BPS configurations with $\mathcal{N} = (1, 1)$ preserved supersymmetry, the symmetry of the corresponding BPS equations is $SU(2) \times SU(2)$ for which Y^a ($a = 1, 2$) and $Y^{\dot{a}}$ ($\dot{a} = 3, 4$) are in $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$, respectively. The BPS equations are given by

$$D_2 Y^a = -\epsilon^{ab} \epsilon_{\dot{c}\dot{d}} \beta_b^{\dot{c}\dot{d}}, \quad D_0 Y^a = D_1 Y^a = 0, \quad (\text{A.33})$$

$$D_2 Y^{\dot{a}} = -\epsilon^{\dot{a}\dot{b}} \epsilon_{cd} \beta_{\dot{b}}^{cd}, \quad D_0 Y^{\dot{a}} = D_1 Y^{\dot{a}} = 0. \quad (\text{A.34})$$

$$\beta_b^{ab} = \beta_b^{a\dot{b}}, \quad \beta_b^{\dot{a}\dot{b}} = \beta_b^{\dot{a}b}. \quad (\text{A.35})$$

1/4 BPS equations

- $\mathcal{N} = (3, 0)$

For the BPS configurations with $\mathcal{N} = (3, 0)$ preserved supersymmetry, the symmetry of the corresponding BPS equations is $SO(4)$ defined by

$$U^T g U = g, \quad U \in SU(4)_R, \quad (\text{A.36})$$

where the $SO(4)$ invariant tensor g is given by

$$g_{AB} \equiv \left(\begin{array}{c|c} & i\sigma_2 \\ \hline -i\sigma_2 & \end{array} \right). \quad (\text{A.37})$$

The scalar fields Y^A ($A = 1, 2, 3, 4$) are in $\mathbf{4}$ of $SO(4)$. The BPS equations are given by

$$D_2 Y^A = -\frac{1}{3}(\epsilon^{ABCD}\beta_{BCD} - \beta_D^{AD}), \quad (D_0 - D_1)Y^A = 0, \quad (\text{A.38})$$

$$\beta_{ABC} = \beta_{[ABC]} + \frac{1}{2}\epsilon_{DE[AB}\beta_{C]}^{DE} - \frac{2}{3}g_{A[B}g_{C]E}\beta_D^{ED} - \frac{1}{3}\epsilon_{BC}^{DE}(\beta_{DAE} + \beta_{ADE}). \quad (\text{A.39})$$

Here, we have used the $SO(4)$ invariant tensor g to lower the indices

$$\beta_{ABC} \equiv g_{BB'}g_{CC'}\beta_A^{B'C'}, \quad \epsilon_{BC}^{DE} \equiv g_{BB'}g_{CC'}\epsilon^{B'C'DE}. \quad (\text{A.40})$$

- $\mathcal{N} = (2, 1)$

For the BPS configurations with $\mathcal{N} = (2, 1)$ preserved supersymmetry, the symmetry of the corresponding BPS equations is $SU(2) \times U(1)$ for which Y^a ($a = 1, 2$) and $Y^{\dot{a}}$ ($\dot{a} = 3, 4$) are in $\mathbf{2}_1$ and $\mathbf{2}_{-1}$, respectively. The BPS equations are given by

$$D_2 Y^a = -\beta_b^{ab} + \beta_{\dot{b}}^{ab}, \quad D_0 Y^a = D_1 Y^a = 0, \quad (\text{A.41})$$

$$D_2 Y^{\dot{a}} = -\beta_{\dot{b}}^{\dot{a}\dot{b}} + \beta_b^{\dot{a}\dot{b}}, \quad D_0 Y^{\dot{a}} = D_1 Y^{\dot{a}} = 0. \quad (\text{A.42})$$

$$\beta_b^{\dot{c}\dot{d}} = \beta_{\dot{b}}^{cd} = 0, \quad \sum_{b=1}^2 \beta_b^{a,b+2} = \sum_{\dot{b}=3}^4 \beta_{\dot{b}}^{a,\dot{b}-2} = 0. \quad (\text{A.43})$$

1/3 BPS equations

- $\mathcal{N} = (4, 0)$

For the BPS configurations with $\mathcal{N} = (4, 0)$ preserved supersymmetry, the symmetry of the corresponding BPS equations is $SU(2) \times SU(2) \times U(1)$ for which Y^α ($\alpha = 1, 3$) and $Y^{\dot{\alpha}}$ ($\dot{\alpha} = 2, 4$) are in $(\mathbf{2}, \mathbf{1})_1$ and $(\mathbf{1}, \mathbf{2})_{-1}$, respectively. The BPS equations are given by

$$D_2 Y^\alpha = \beta_\beta^{\alpha\beta}, \quad (D_0 - D_1)Y^\alpha = 0, \quad (\text{A.44})$$

$$D_2 Y^{\dot{\alpha}} = \beta_{\dot{\beta}}^{\dot{\alpha}\dot{\beta}}, \quad (D_0 - D_1)Y^{\dot{\alpha}} = 0. \quad (\text{A.45})$$

$$\beta_\alpha^{\dot{\alpha}\beta} = \frac{1}{2}\delta_\alpha^\beta \beta_{\dot{\gamma}}^{\dot{\alpha}\gamma}, \quad \beta_{\dot{\alpha}}^{\alpha\dot{\beta}} = \frac{1}{2}\delta_{\dot{\alpha}}^{\dot{\beta}} \beta_{\dot{\gamma}}^{\alpha\gamma}. \quad (\text{A.46})$$

- $\mathcal{N} = (3, 1)$

For the BPS configurations with $\mathcal{N} = (3, 1)$ preserved supersymmetry, the symmetry of the corresponding BPS equations is $SU(2) \times U(1)$ for which Y^α ($\alpha = 1, 3$) and $Y^{\dot{\alpha}}$ ($\dot{\alpha} = 2, 4$) are in $\mathbf{2}_1$ and $\mathbf{2}_{-1}$, respectively. The BPS equations are given by

$$D_2 Y^\alpha = 2\beta_\beta^{\alpha\beta}, \quad D_0 Y^\alpha = D_1 Y^\alpha = 0, \quad (\text{A.47})$$

$$D_2 Y^{\dot{\alpha}} = 2\beta_{\dot{\beta}}^{\dot{\alpha}\dot{\beta}}, \quad D_0 Y^{\dot{\alpha}} = D_1 Y^{\dot{\alpha}} = 0. \quad (\text{A.48})$$

$$\beta_\alpha^{\dot{\beta}\beta} = \frac{1}{2}\delta_\alpha^\beta(\beta_{\dot{\gamma}}^{\dot{\beta}\gamma} + \beta_{\dot{\gamma}}^{\dot{\beta}\gamma}) - \delta_\alpha^{\dot{\beta}-1}\beta_{\dot{\gamma}}^{\beta+1,\dot{\gamma}}, \quad (\text{A.49})$$

$$\beta_{\dot{\alpha}}^{\beta\dot{\beta}} = \frac{1}{2}\delta_{\dot{\alpha}}^{\dot{\beta}}(\beta_{\dot{\gamma}}^{\beta\dot{\gamma}} + \beta_{\dot{\gamma}}^{\beta\dot{\gamma}}) - \delta_{\dot{\alpha}}^{\beta+1}\beta_{\dot{\gamma}}^{\dot{\beta}-1,\gamma}. \quad (\text{A.50})$$

- $\mathcal{N} = (2, 2)$

For the BPS configurations with $\mathcal{N} = (2, 2)$ preserved supersymmetry, the symmetry of the BPS equations is $U(1)^3$ corresponding to the Cartan subalgebra of $SU(4)_R$. The BPS equations are given by

$$D_2 Y^a = -(\beta_b^{ab} - \beta_{\dot{b}}^{a\dot{b}}), \quad D_0 Y^a = D_1 Y^a = 0, \quad (\text{A.51})$$

$$D_2 Y^{\dot{a}} = -(\beta_{\dot{b}}^{\dot{a}\dot{b}} - \beta_b^{\dot{a}b}), \quad D_0 Y^{\dot{a}} = D_1 Y^{\dot{a}} = 0, \quad (\text{A.52})$$

$$\beta_b^{\dot{a}\dot{b}} = \beta_{\dot{b}}^{ab} = 0, \quad \beta_1^{a3} = \beta_2^{a4} = \beta_3^{\dot{a}1} = \beta_4^{\dot{a}2} = 0, \quad (\text{A.53})$$

where the indices run as $a = 1, 2, \dot{a} = 3, 4$.

5/12 BPS equations

- $\mathcal{N} = (5, 0)$

The matrix Ξ for the 1/12 BPS configurations is invariant under $SO(5)$, so that the corresponding BPS equations have $Spin(5) \cong USp(4)$ symmetry defined by

$$U^T \Gamma_6 U = \Gamma_6, \quad U \in SU(4). \quad (\text{A.54})$$

The scalar fields Y^A ($A = 1, 2, 3, 4$) are in $\mathbf{4}$ of $USp(4)$. The BPS equations are given by

$$D_2 Y^A = \frac{1}{5} \left(\beta_D^{AD} - (\Gamma_6^\dagger)^{AD} (\Gamma_6)_{BC} \beta_D^{BC} \right), \quad (D_0 - D_1) Y^A = 0. \quad (\text{A.55})$$

$$\beta_A^{BC} = -\frac{1}{4} \left[\delta_A^D (\Gamma_6^\dagger)^{BC} (\Gamma_6)_{EF} + 2(\Gamma_6)_{AF} (\Gamma_6^\dagger)^{D[B} \delta_E^{C]} + 2\delta_A^{[B} \delta_E^{C]} \delta_F^D \right] \beta_D^{EF}. \quad (\text{A.56})$$

- $\mathcal{N} = (4, 1)$

For the BPS configurations with $\mathcal{N} = (4, 1)$ preserved supersymmetry, the symmetry of the corresponding BPS equations is $SU(2) \times SU(2)$ for which Y^α ($\alpha = 1, 3$) and $Y^{\dot{\alpha}}$ ($\dot{\alpha} = 2, 4$) are in $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$, respectively. The BPS equations are given by

$$D_2 Y^\alpha = \beta_\beta^{\alpha\beta}, \quad D_0 Y^\alpha = D_1 Y^\alpha = 0, \quad (\text{A.57})$$

$$D_2 Y^{\dot{\alpha}} = \beta_{\dot{\beta}}^{\dot{\alpha}\dot{\beta}}, \quad D_0 Y^{\dot{\alpha}} = D_1 Y^{\dot{\alpha}} = 0. \quad (\text{A.58})$$

$$\beta_\alpha^{\dot{\alpha}\dot{\beta}} = \epsilon_{\alpha\beta} \epsilon^{\dot{\gamma}[\dot{\alpha}} \beta_{\dot{\gamma}}^{\dot{\beta}]\beta}, \quad \beta_{\dot{\alpha}}^{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{\gamma[\alpha} \beta_\gamma^{\beta]\dot{\beta}}, \quad (\text{A.59})$$

- $\mathcal{N} = (3, 2)$

For the BPS configurations with $\mathcal{N} = (3, 2)$ preserved supersymmetry, the corresponding BPS equations have $SU(2)$ symmetry for which both Y^α ($\alpha = 1, 3$) and $Y^{\dot{\alpha}}$ ($\dot{\alpha} = 2, 4$) are in $\mathbf{2}$ and $SO(2)$ symmetry which rotates these two doublets. The BPS equations are given by

$$D_2 Y^\alpha = 2\beta_\beta^{\alpha\beta}, \quad D_0 Y^\alpha = D_1 Y^\alpha = 0, \quad (\text{A.60})$$

$$D_2 Y^{\dot{\alpha}} = 2\beta_{\dot{\beta}}^{\dot{\alpha}\dot{\beta}}, \quad D_0 Y^{\dot{\alpha}} = D_1 Y^{\dot{\alpha}} = 0. \quad (\text{A.61})$$

$$\beta_\alpha^{\dot{\beta}\beta} = \frac{1}{2} \delta_\alpha^\beta (\beta_\gamma^{\dot{\beta}\gamma} + \beta_{\dot{\gamma}}^{\dot{\beta}\gamma}) - \delta_\alpha^{\dot{\beta}-1} \beta_{\dot{\gamma}}^{\beta+1, \dot{\gamma}}, \quad (\text{A.62})$$

$$\beta_{\dot{\alpha}}^{\beta\dot{\beta}} = \frac{1}{2} \delta_{\dot{\alpha}}^{\dot{\beta}} (\beta_\gamma^{\beta\dot{\gamma}} + \beta_\gamma^{\beta\gamma}) - \delta_{\dot{\alpha}}^{\beta+1} \beta_\gamma^{\dot{\beta}-1, \gamma}, \quad (\text{A.63})$$

$$\epsilon_{\dot{\alpha}\dot{\beta}} \beta_\alpha^{\dot{\alpha}\dot{\beta}} = \epsilon_{\alpha\beta} (\beta_\gamma^{\beta\gamma} + \beta_{\dot{\gamma}}^{\beta\dot{\gamma}}), \quad \epsilon_{\alpha\beta} \beta_{\dot{\alpha}}^{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} (\beta_\gamma^{\dot{\beta}\gamma} + \beta_{\dot{\gamma}}^{\dot{\beta}\dot{\gamma}}). \quad (\text{A.64})$$

1/2 BPS equations

- $\mathcal{N} = (3, 3)$

For the BPS configurations with $\mathcal{N} = (3, 3)$ preserved supersymmetry, the symmetry of the corresponding BPS equations is $SO(4)$ defined by

$$U^T g U = g, \quad U \in SU(4)_R, \quad (\text{A.65})$$

where the $SO(4)$ invariant tensor g is given by

$$g \equiv \left(\begin{array}{c|c} & i\sigma_2 \\ \hline -i\sigma_2 & \end{array} \right). \quad (\text{A.66})$$

The scalar fields Y^A ($A = 1, 2, 3, 4$) are in $\mathbf{4}$ of $SO(4)$. The BPS equations are given by

$$D_2 Y^A = -\frac{1}{3} \epsilon^{ABCD} \beta_{BCD}, \quad D_0 Y^A = D_1 Y^A = 0. \quad (\text{A.67})$$

$$\beta_{ABC} = \beta_{[ABC]}, \quad (\text{A.68})$$

where we have lowered the indices by using the $SO(4)$ invariant tensor g as

$$\beta_{ABC} \equiv g_{BB'} g_{CC'} \beta_A^{B'C'}. \quad (\text{A.69})$$

- $\mathcal{N} = (4, 2)$

For the BPS configurations with $\mathcal{N} = (4, 2)$ preserved supersymmetry, the symmetry of the corresponding BPS equations is $SU(2) \times SU(2) \times U(1)$ for which Y^α ($\alpha = 1, 3$) and $Y^{\dot{\alpha}}$ ($\dot{\alpha} = 2, 4$) are in $(\mathbf{2}, \mathbf{1})_1$ and $(\mathbf{1}, \mathbf{2})_{-1}$, respectively. The BPS equations are given by

$$D_2 Y^\alpha = \beta_\beta^{\alpha\beta}, \quad D_0 Y^\alpha = D_1 Y^\alpha = 0, \quad (\text{A.70})$$

$$D_2 Y^{\dot{\alpha}} = \beta_{\dot{\beta}}^{\dot{\alpha}\dot{\beta}}, \quad D_0 Y^{\dot{\alpha}} = D_1 Y^{\dot{\alpha}} = 0, \quad (\text{A.71})$$

$$\beta_\alpha^{\beta\dot{\alpha}} = \beta_{\dot{\alpha}}^{\dot{\beta}\alpha} = \beta_{\dot{\alpha}}^{\alpha\beta} = \beta_\alpha^{\dot{\alpha}\dot{\beta}} = 0. \quad (\text{A.72})$$

- $\mathcal{N} = (5, 1)$

The matrix Ξ for the 1/12 BPS configurations is invariant under $SO(5)$, so that the corresponding BPS equations have $Spin(5) \cong USp(4)$ symmetry defined by

$$U^T \Gamma_6 U = \Gamma_6, \quad U \in SU(4). \quad (\text{A.73})$$

The scalar fields Y^A ($A = 1, 2, 3, 4$) are in $\mathbf{4}$ of $USp(4)$. The BPS equations are given by

$$D_2 Y^A = \frac{1}{2} \beta_D^{AD}, \quad D_0 Y^A = D_1 Y^A = 0. \quad (\text{A.74})$$

$$\beta_A^{BC} = \frac{1}{4} \left[(\Gamma_6^\dagger)^{BC} (\Gamma_6)_{AD} - 2 \delta_A^{[B} \delta_D^{C]} \right] \beta_E^{DE}. \quad (\text{A.75})$$

- $\mathcal{N} = (6, 0)$

Since the matrix Ξ for $\mathcal{N} = (6, 0)$ BPS configurations is invariant under $SO(6)_R$, the corresponding BPS equations have $SU(4)_R$ symmetry for which the scalar fields Y^A ($A = 1, 2, 3, 4$) are in the fundamental representation. The BPS equations are given by

$$D_2 Y^A = \frac{1}{3} \beta_B^{AB}, \quad (D_0 - D_1) Y^A = 0. \quad (\text{A.76})$$

$$\beta_A^{BC} = -\frac{2}{3} \delta_A^{[B} \beta_D^{C]D}. \quad (\text{A.77})$$

A.3 M2-branes and M5-branes

For the BPS configurations with M2-branes and M5-branes, the preserved supercharges are specified by the conditions of the form

$$\gamma_2 \Xi_{ij} \epsilon_j = \epsilon_i, \quad \gamma_1 \Xi'_{ij} \epsilon_j = \epsilon_i, \quad (\text{A.78})$$

where Ξ and Ξ' are 6-by-6 real symmetric matrices satisfying $\Xi^2 = \Xi'^2 = \mathbf{1}_6$. The BPS equations below preserve n ($n = 1, 2, 3$) supercharges and specified by the following matrices

$$\Xi = \text{diag}(\sigma_3, *, *) , \quad \Xi' = \text{diag}(\sigma_1, *, *) \quad \text{for } n = 1, \quad (\text{A.79})$$

$$\Xi = \text{diag}(\sigma_3, \sigma_3, *) , \quad \Xi' = \text{diag}(\sigma_1, \sigma_1, *) \quad \text{for } n = 2, \quad (\text{A.80})$$

$$\Xi = \text{diag}(\sigma_3, \sigma_3, \sigma_3) , \quad \Xi' = \text{diag}(\sigma_1, \sigma_1, \sigma_1) \quad \text{for } n = 3. \quad (\text{A.81})$$

The explicit form of the BPS equations is given as follows,

$$-D_0 Y^B (\Gamma_j)_{BA} (\Xi \Xi')_{ji} + D_1 Y^B (\Gamma_j)_{BA} \Xi'_{ji} + D_2 Y^B (\Gamma_j)_{BA} \Xi_{ji} + \Upsilon_A^{BC} (\Gamma_i)_{BC} = 0, \quad (\text{A.82})$$

where $1 \leq i, j \leq 2n$. The BPS bound for energy is given by

$$\begin{aligned} E \geq & \frac{k}{4\pi n} \int d^2x \epsilon^{MN} \partial_M \text{Tr} \left[Y_A^\dagger D_N Y^B \right] (\Gamma_i \Gamma_j^\dagger)_B{}^A (\Xi \Xi')_{ji} \\ & + \frac{k}{8\pi n} \int d^2x \partial_1 \text{Tr} \left[Y_A^\dagger \Upsilon_D^{BC} \right] (\Gamma_i)_{BC} (\Gamma_j^\dagger)^{AD} \Xi'_{ij} \\ & + \frac{k}{8\pi n} \int d^2x \partial_2 \text{Tr} \left[Y_A^\dagger \Upsilon_D^{BC} \right] (\Gamma_i)_{BC} (\Gamma_j^\dagger)^{AD} \Xi_{ij}, \end{aligned} \quad (\text{A.83})$$

and reduces to the following form for the BPS configurations

$$\begin{aligned} E = & \frac{k}{8\pi n} \int d^2x \epsilon^{MN} \partial_M \text{Tr} \left[Y_A^\dagger D_N Y^B \right] (\Gamma_i \Gamma_j^\dagger)_B{}^A (\Xi \Xi')_{ji} \\ & + \frac{k}{4\pi} \int d^2x \partial_M \text{Tr} \left[Y_A^\dagger D^M Y^A \right]. \end{aligned} \quad (\text{A.84})$$

- 1/12 BPS equations

For the 1/12 BPS configurations with matrices

$$\Xi = \left(\begin{array}{c|c} \sigma_3 & \\ \hline & *_4 \end{array} \right), \quad \Xi' = \left(\begin{array}{c|c} \sigma_1 & \\ \hline & *_4 \end{array} \right), \quad (\text{A.85})$$

the symmetry of the BPS equations is $SU(2) \times SU(2)$ for which Y^a ($a = 1, 2$) and $Y^{\dot{a}}$ ($\dot{a} = 3, 4$) are in $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$, respectively. The BPS equations are given by

$$D_0 Y^a = -i(\beta_d^{ad} - \beta_{\dot{d}}^{a\dot{d}}), \quad (D_1 + iD_2) Y^a = -i\epsilon^{ad} \epsilon_{b\dot{c}} \beta_d^{b\dot{c}}, \quad (\text{A.86})$$

$$D_0 Y^{\dot{a}} = -i(\beta_d^{\dot{a}d} - \beta_{\dot{d}}^{\dot{a}\dot{d}}), \quad (D_1 - iD_2) Y^{\dot{a}} = i\epsilon^{\dot{a}d} \epsilon_{bc} \beta_{\dot{d}}^{bc}. \quad (\text{A.87})$$

- 1/6 BPS equations

For the 1/6 BPS configurations with matrices

$$\Xi = \left(\begin{array}{c|c|c} \sigma_3 & & \\ \hline & \sigma_3 & \\ \hline & & * _2 \end{array} \right), \quad \Xi' = \left(\begin{array}{c|c|c} \sigma_1 & & \\ \hline & \sigma_1 & \\ \hline & & * _2 \end{array} \right), \quad (\text{A.88})$$

the BPS equations have $U(1)^2$ symmetry, for one of which Y^α ($\alpha = 1, 3$) and $Y^{\dot{\alpha}}$ ($\dot{\alpha} = 2, 4$) have plus and minus charges. The other $U(1)$ transformation is defined by

$$Y^1 \rightarrow \cos \theta Y^1 + i \sin \theta Y^3, \quad Y^3 \rightarrow i \sin \theta Y^1 + \cos \theta Y^3. \quad (\text{A.89})$$

The BPS equations are given by

$$D_0 Y^a = -i(\beta_d^{ad} - \beta_d^{\dot{a}d}), \quad (D_1 + iD_2)Y^a = -i\epsilon^{ad}\epsilon_{\dot{b}\dot{c}}\beta_d^{bc}, \quad (\text{A.90})$$

$$D_0 Y^{\dot{a}} = -i(\beta_d^{\dot{a}d} - \beta_d^{ad}), \quad (D_1 - iD_2)Y^{\dot{a}} = i\epsilon^{\dot{a}d}\epsilon_{bc}\beta_d^{bc}. \quad (\text{A.91})$$

$$(D_1 - iD_2)Y^1 = 2i\beta_4^{32}, \quad (D_1 + iD_2)Y^3 = 2i\beta_2^{41}, \quad (\text{A.92})$$

$$\beta_1^{A1} = \beta_3^{A3}, \quad \beta_1^{A3} = \beta_3^{A1}. \quad (\text{A.93})$$

- 1/4 BPS equations

For the 1/4 BPS configurations with matrices

$$\Xi = \left(\begin{array}{c|c|c} \sigma_3 & & \\ \hline & \sigma_3 & \\ \hline & & \sigma_3 \end{array} \right), \quad \Xi' = \left(\begin{array}{c|c|c} \sigma_1 & & \\ \hline & \sigma_1 & \\ \hline & & \sigma_1 \end{array} \right), \quad (\text{A.94})$$

the symmetry of the BPS equations is $SO(3)$ defined by

$$U^T g U = g, \quad U \in SU(3), \quad g \equiv \text{diag}(-1, 1, 1). \quad (\text{A.95})$$

The scalar fields Y^i ($i = 1, 2, 3$) and Y^4 are in **3** and **1** of $SO(3)$ symmetry. The BPS equations are given by

$$D_0 Y^i = i\beta_4^{i4}, \quad (D_1 + iD_2)Y^i = \frac{i}{3}\epsilon^{ijk}\beta_{j4k}, \quad (D_1 - iD_2)Y^i = -i\epsilon^{ijk}\beta_{4jk}, \quad (\text{A.96})$$

$$D_0 Y^4 = -\frac{i}{3}\beta_i^{4i}, \quad (D_1 - iD_2)Y^4 = \frac{i}{3}\epsilon^{ijk}\beta_{ijk}. \quad (\text{A.97})$$

$$\beta_{ijk} = \beta_{[ijk]}, \quad \beta_{i4j} + \beta_{j4i} = \frac{2}{3}g_{ij}\beta_k^{4k}. \quad (\text{A.98})$$

Here, we have used the $SO(3)$ invariant tensor g to lower the indices

$$\beta_{ijk} = g_{jj'}g_{kk'}\beta_i^{j'k'}, \quad \beta_{i4j} \equiv g_{jj'}\beta_i^{4j}. \quad (\text{A.99})$$

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